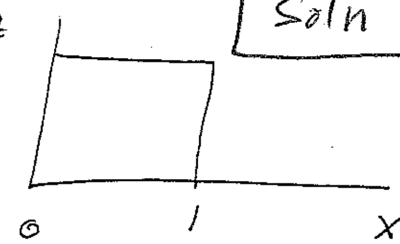
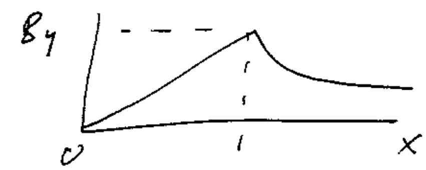
Δ' from Constant J_z in Slab

Tearing Mode — Outer Soln

$$\tilde{\psi}'' = k^2\tilde{\psi} + \frac{k_{\parallel}''}{k_{\parallel}}\tilde{\psi}$$



$$\frac{k_{\parallel}''}{k_{\parallel}} = \frac{B_y''}{B_y} = \frac{J_z'}{B_y}$$



$x > 0$: $J_z = J_1 H(1-x)$; $J_z' = -J_1\,\delta(1-x)$

$$B_y(1) \equiv J_1 \qquad \boxed{a = 1}$$

Jump condition: $[\tilde{\psi}']_{x=1} = -\tilde{\psi}$ —— ①

let $\tilde{\psi} = \begin{cases} \exp[-k(x-1)] & ; x > 1 \\ A\dfrac{\cosh kx}{\cosh k} + B\dfrac{\sinh kx}{\sinh k} & ; x < 1 \end{cases}$

$A + B = 1$ —— ②

$$\Delta' \equiv \left.\frac{\tilde{\psi}'}{\tilde{\psi}}\right|_{x\to 0} = k\,\frac{B}{A}\tanh^{-1}k \quad \text{—— ③}$$

From ①, $1 + (A\tanh k + B\tanh^{-1}k) = 1/k$ —— ④

~~use ② in ④ and ③~~ ② and ④ $\Rightarrow$ A and B

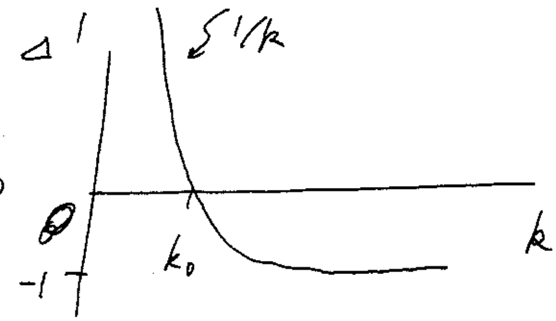
use in ③ to get $\Delta'(k)$

$$\Rightarrow \boxed{\frac{\Delta'}{k} = -\left[\frac{(1/k - 1) - \tanh k}{(1/k-1)\tanh k - 1}\right]} \quad k \geq 0$$

$k \to \infty \qquad \Delta'/k \to -1$

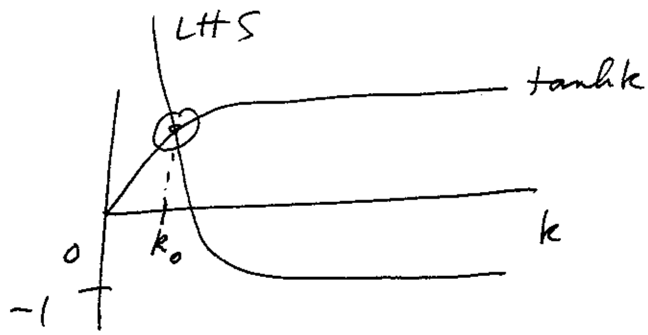
$k \to 0 \qquad \Delta'/k \to 1/k^2$

$\Rightarrow$



Marginal stability:

$$\frac{1}{k_0} - 1 = \tanh k_0$$



T1

<u>TM: inner layer</u>

Tearing Mode – Inner Soln

$$\gamma n M \nabla^2 \tilde{\varphi} = B \cdot \nabla \nabla^2 \tilde{\psi} \quad (A)$$

$$\gamma \tilde{\psi} = \vec{B} \cdot \vec{\nabla} \tilde{\varphi} + \eta \nabla^2 \tilde{\psi} \quad (B) \qquad \{e^{\gamma t}\}$$

(1) $\vec{B} \cdot \vec{\nabla} \rightarrow B_0 \partial_z + B_y \partial_y \rightarrow B_y \partial_y = i k B_y$

let $\boxed{k B_y \equiv k_\parallel B_0}$, $B_y(x) \Rightarrow k_\parallel(x)$

(2) let $\boxed{\tilde{\psi} \rightarrow B_0 \tilde{\psi}}$

(3) Note that in the layer, $x \ll a$;

if $ka \sim 1 \Rightarrow kx \ll 1 \Rightarrow \nabla^2 \rightarrow \partial_x^2$

$$\Rightarrow \gamma n M \tilde{\varphi}'' = i k_\parallel B_0^2 \tilde{\psi}'' \quad (A') \qquad e^{iky}$$

$$\gamma \tilde{\psi} = i k_\parallel \tilde{\varphi} + \eta \tilde{\psi}'' \quad (B')$$

(4) let $\boxed{\frac{B_0^2}{nM} = V_A^2 = 1}$ velocity Normalization

(5) Note, $x \rightarrow 0 \Rightarrow k_\parallel(x) \rightarrow k_\parallel' x$.

$k_\parallel'$ has dimensions of $\frac{1}{L^2}$. Let $\boxed{k_\parallel' = 1}$

$\Rightarrow$ space normalization

$\therefore$ (A') (B') $\rightarrow$

$$\boxed{\begin{aligned} \gamma \tilde{\varphi}'' &= -x \tilde{\psi}'' \\ \gamma \tilde{\psi} &= x \tilde{\varphi} + \eta \tilde{\psi}'' \end{aligned}}$$

let $i\tilde{\varphi} \rightarrow \tilde{\varphi}$

Drop $\tilde{\ }$'s, re write upon substitution as

$$\boxed{\begin{aligned} \gamma \tilde{\varphi}'' &= \frac{x}{\eta}\left(x \tilde{\varphi} - \gamma \tilde{\psi}\right) \quad (C) \\ \eta \tilde{\psi}'' &= \left(\gamma \tilde{\psi} - x \tilde{\varphi}\right) \quad (D) \end{aligned}}$$

inner eqns

η is normalized and $\eta \ll 1$ $(\eta \sim 10^{-8})$

$\therefore$ "ideal MHD" $\Rightarrow$ $\gamma \tilde{\psi} - x \tilde{\varphi} \simeq 0$

Under what conditions do (C), (D) fail,
ie, when does ideal MHD fail?

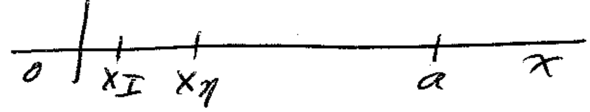
In (C), LHS is small if $\gamma\eta \ll x^4$
In (D), " " if $\eta \ll \gamma x^2$

2 possible x scales

$$x_\eta^{\,2} \equiv \eta/\gamma, \quad x_I^{\,4} \equiv \gamma\eta$$

We will show self consistently that $x_I \ll x_\eta$

ie,



$\therefore$ First possible failure for $x \sim x_\eta \gg x_I$

blow up

$$\Rightarrow \quad 0 \simeq x\varphi - \gamma\psi,$$

$$\eta\psi'' = \gamma\psi - x\varphi$$

$$\Rightarrow \quad \psi'' = 0 \Rightarrow \psi = Ax + B \quad \text{and} \quad \varphi = \frac{\gamma\psi}{x}$$

But this is just the outer equations for $x \ll a$. $\therefore$ This ~~layer~~ scale does not resolve the boundary layer.

Now suppose $x_I \sim x \ll x_\eta$.

$$\Rightarrow \quad \eta\gamma\varphi'' = x(x\varphi - \gamma\psi)$$

$$\eta\psi'' = 0$$

Now this seems viable since φ is well behaved for $x \to 0$.

Solve these:

$$\Rightarrow \quad \boxed{\psi = A} = \text{const} \quad (\text{even solns})$$

("constant ψ approximation")

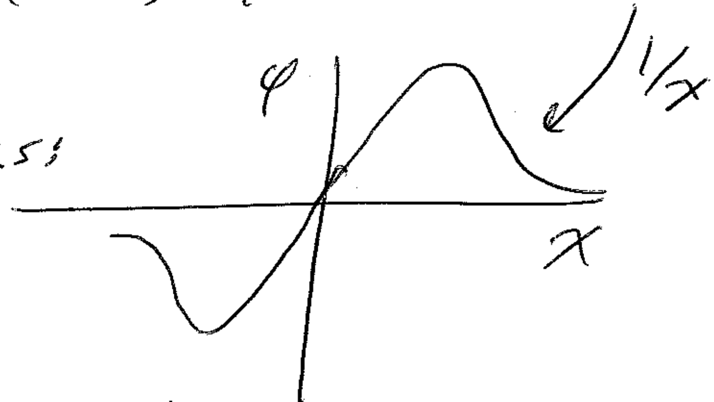
$$\Rightarrow \quad \boxed{\eta\gamma\varphi'' = x(x\varphi - \gamma A)}$$

inhomogeneous eqn for $\varphi(x)$.

T4

The φ eqn can be solved.

φ must be odd $\Rightarrow \varphi(0) = 0.$

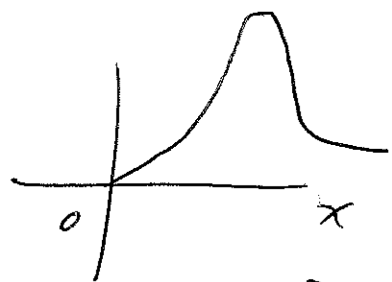


Asymptotic behaviours:

$$x \to \infty \Rightarrow \varphi \to \frac{\gamma A}{x}$$

$$x \to 0 \Rightarrow \eta\cancel{\gamma}\varphi'' \to -x\cancel{\gamma}A$$

$$\Rightarrow \varphi \to Cx - \frac{Ax^3}{\eta 6}$$

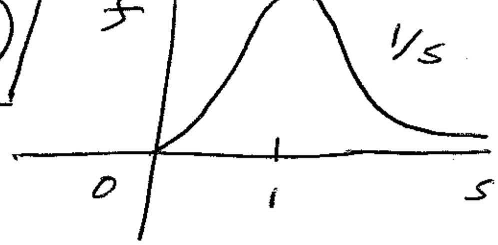


The full soln is a Parabolic Cylinder Fn. Will not do here. But we can normalize. let $\boxed{\varphi \equiv \frac{\gamma A}{\Delta} f}$

where $\boxed{\gamma\eta \equiv \Delta^4}$ let $\boxed{x = \Delta s}$

$$\Rightarrow \boxed{f_{ss} = s(sf - 1)}$$

$\Rightarrow$



Thus φ is known in terms of A.

Now ψ is needed to first order for asymptotic matching

$$\eta \psi_0'' = 0 \implies \psi_0 = A$$

$$\eta \psi_1'' = \gamma A - x\varphi$$

$$\implies \eta \psi_1'(x) = \int_0^x dx\,(\gamma A - x\varphi)$$

Matching

$$\psi_{in}(x\to\infty) \longleftrightarrow \psi_{out}(x\to 0)$$

$$\psi_{in}'(x\to\infty) \longleftrightarrow \psi_{out}'(x\to 0)$$

$$\eta \psi_{in}'(x\to\infty) = \int_0^\infty dx\,(\gamma A - x\varphi)$$

$$\implies \eta \psi_{in}'(\infty) = \gamma A \Delta \int_0^\infty ds\,(1-sf)$$

let $\psi_{out}'(x\to 0) \implies \Delta' \psi_{out}(0)$

$$\implies \eta \Delta' = \gamma \Delta \int_0^\infty ds\,(1-sf)$$

But $\boxed{\int_0^\infty ds\,(1-sf) = \# \equiv I}$

$$\implies \boxed{\gamma = \frac{\eta \Delta'}{I \Delta}} \qquad \boxed{\gamma\eta \equiv \Delta^4}$$

$$\Rightarrow \boxed{\gamma = \eta^{3/5} \Delta'^{4/5} / I^{4/5}}$$

Normalized to V_A and $k_{\parallel}'$.

<u>Note:</u> $\gamma \sim \eta^{3/5} \Rightarrow \boxed{\eta \ll \gamma \ll 1}$

<u>In unnormalized units:</u>

$$\boxed{\gamma = \eta^{3/5} (k_{\parallel}' V_A)^{2/5} \left(\frac{\Delta'}{I}\right)^{4/5}}$$

$$\boxed{\Delta = \frac{\eta^{2/5} \left(\frac{\Delta'}{I}\right)^{1/5}}{k_{\parallel}'^{2/5} V_A^{2/5}}}$$

$\gamma : \frac{1}{\tau_A} \sim 1 : S^{3/5}$ $\qquad \boxed{S \equiv \frac{\tau_R}{\tau_A}}$

$\Delta : a \sim 1 : S^{2/5}$