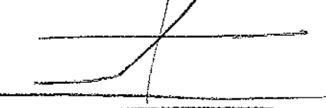
KH for  profile

$$(\omega - ku)\nabla^2\tilde{\varphi} = -ku''\tilde{\varphi}\ , \quad u(x)$$

parity: $x \to -x \Rightarrow \nabla^2 \to \nabla^2,\ u'' \to u''$

For even u, $u \to u \Rightarrow$ same eqn

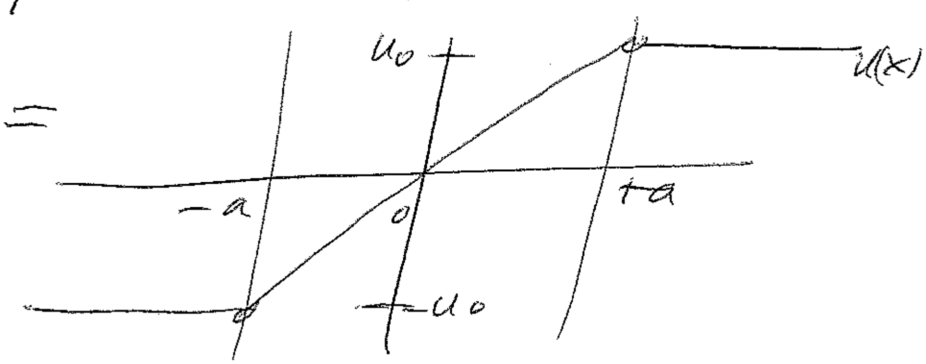
$\therefore\ \varphi(x)$ is a soln $\Rightarrow \varphi(-x)$ is a soln

$\Rightarrow$ even & odd parities to soln, ie,

$$\varphi_e = \varphi(x) + \varphi(-x)$$
$$\varphi_o = \varphi(x) - \varphi(-x)$$

For odd u, $u \to -u \Rightarrow \varphi(-x)$ satisfies a different eqn. $\Rightarrow$ No simple parities

Suppose $u(x) =$



$u'' = 0$ except at $|x| = a$.

Thus,
$\underline{x > a} \Rightarrow \tilde{\varphi}_> = e^{-k(x-a)}$

$\underline{x < -a} \Rightarrow \tilde{\varphi}_< = Ce^{k(x+a)}$

$\underline{|x| < a} \Rightarrow \tilde{\varphi} = A_+e^{kx} + A_-e^{-kx}$

cd1 ~~[illegible]~~

<u>Jump conditions</u> at $|x| = a$.

$$\bar{\omega}\breve{\varphi}'' = \bar{\omega}''\breve{\varphi} \quad , \text{ let } \bar{\omega} \equiv \omega - ku(x)$$

$$\Rightarrow (\bar{\omega}\breve{\varphi}')' = (\bar{\omega}'\breve{\varphi})'$$

$$\Rightarrow [\bar{\omega}\breve{\varphi}'] = [\bar{\omega}'\breve{\varphi}] \text{ —— ①}$$

$$\Rightarrow \bar{\omega}\breve{\varphi}' = \bar{\omega}'\breve{\varphi} + c$$

$$\Rightarrow \left(\frac{\breve{\varphi}}{\bar{\omega}}\right)' = \frac{c}{\bar{\omega}^2} \Rightarrow \left[\frac{\breve{\varphi}}{\bar{\omega}}\right] = 0 \text{ —— ②}$$

$$[f]_-^+ \equiv [f]_{a-\epsilon}^{a+\epsilon} \ , \text{ or, } [f]_{-a-\epsilon}^{-a+\epsilon}$$

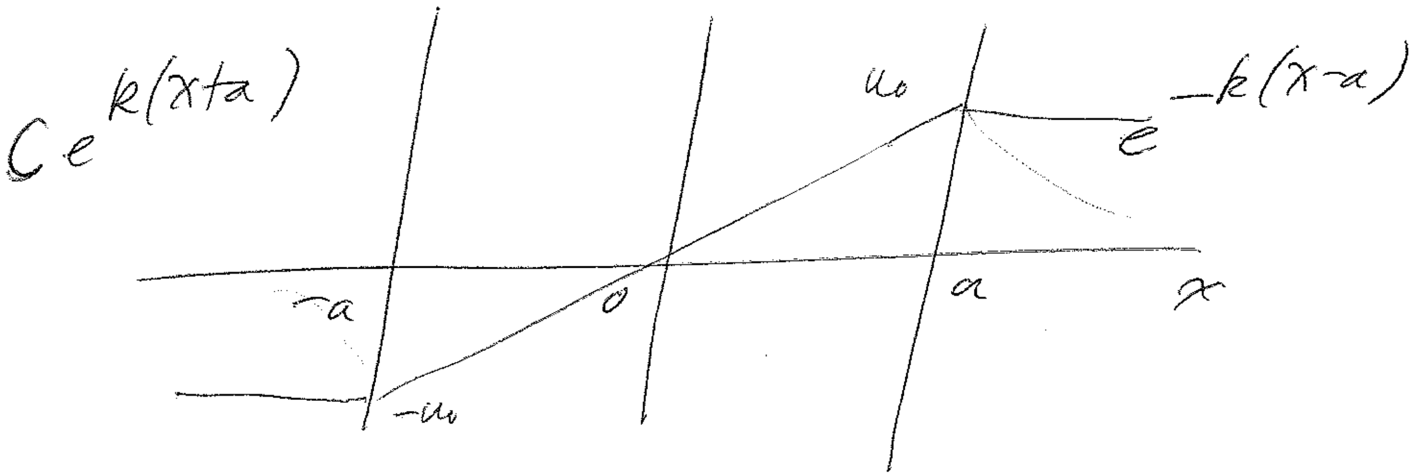
Now $[\bar{\omega}]_-^+ = 0$

Then, ② $\Rightarrow \boxed{[\breve{\varphi}]_-^+ = 0}$

using this in ① $\Rightarrow \boxed{\bar{\omega}[\breve{\varphi}']_-^+ = \breve{\varphi}[\bar{\omega}']}$

2 jumps ↑

Algebra for Roots of KH



$$A_+ e^{kx} + A_- e^{-kx}$$

$$[\tilde{\varphi}] = 0 , \quad \bar{\omega}[\tilde{\varphi}'] = \tilde{\varphi}[\bar{\omega}']$$

$$\bar{\omega} = \omega - k u_0 , \quad u_0' = u_0 / a$$

$$[\bar{\omega}']_a = -k[u']_a = +k u_0'$$

$$[\bar{\omega}']_{-a} = -k[u']_{-a} = -k u_0'$$

$$\bar{\omega}_+ = \omega - k u_0 , \quad \bar{\omega}_- = \omega + k u_0$$

$\underline{x = a}$ $$\boxed{1 = A_+ e^{ka} + A_- e^{-ka}} \quad (a)$$

$$+k + kA_+ e^{ka} - kA_- e^{-ka} = -k u_0' / \bar{\omega}_+$$

$$\Rightarrow \boxed{1 + A_+ e^{ka} - A_- e^{-ka} = -\cancel{k} u_0' / \bar{\omega}_+} \quad (b)$$

$x = -a$

$$\boxed{C = A_+ e^{-ka} + A_- e^{ka}}$$ (c)

$$-kA_+ e^{-ka} + kA_- e^{ka} + Ck = C\,k u_0'/\bar{\omega}_-$$

$$\Rightarrow \boxed{C - A_+ e^{-ka} + A_- e^{ka} = C u_0'/\bar{\omega}_-}$$ (d)

$\{a, b, c, d\}$ 4 eqns for $\{A_+, A_-, C, \omega\}$

(a) → (b) ⇒

$$\boxed{2A_+ e^{ka} = -u_0'/\bar{\omega}_+}$$ (e)

(c) → (d) ⇒

$$\boxed{2A_- e^{ka} = C u_0'/\bar{\omega}_-}$$

(e1) + (e2) $\longrightarrow$ (a)

$$\Rightarrow \boxed{\cancel{\#} = -\frac{u_0'}{2\bar{\omega}_+} + \frac{C u_0'}{2\bar{\omega}_-} e^{-2ka}} \quad (f1)$$

(e1) + (e2) $\longrightarrow$ (c)

$$\boxed{C = -\frac{u_0'}{2\bar{\omega}_+} e^{-2ka} + \frac{C u_0'}{2\bar{\omega}_-}} \quad (f2)$$

(f1) $\rightarrow$ $$\left(1 + \frac{u_0'}{2\bar{\omega}_+}\right) = \frac{C u_0'}{2\bar{\omega}_-} e^{-2ka}$$

(f2) $\rightarrow$ $$C\left(1 - \frac{u_0'}{2\bar{\omega}_-}\right) = -\frac{u_0'}{2\bar{\omega}_+} e^{-2ka}$$

(f1) $\times$ (f2) $\Rightarrow$ $$\boxed{\left(1 + \frac{u_0'}{2\bar{\omega}_+}\right)\left(1 - \frac{u_0'}{2\bar{\omega}_-}\right) = -\left(\frac{u_0'}{2}\right)^2 \frac{e^{-4ka}}{\bar{\omega}_+ \bar{\omega}_+}}$$

d.r.

first, chk lim $ka \to 0$.

$$(1-4ka+8k^2a^2)$$

$$\Rightarrow \left(1+\frac{u_0'}{2\bar\omega_+}\right)\left(1-\frac{u_0'}{2\bar\omega_-}\right) \cong -\left(\frac{u_0'}{2}\right)^2 \frac{1}{\bar\omega_+\bar\omega_-}$$

$$\cancel{1-\frac{u_0'}{2\bar\omega_-}+\frac{u_0'}{2\bar\omega_+}-\left(\frac{u_0'}{2}\right)^2\frac{1}{\bar\omega_+\bar\omega_-}}$$

$$\cong -\left(\frac{u_0'}{2}\right)^2\frac{1}{\bar\omega_+\bar\omega_-}\left(\cancel{1}-4ka+8k^2a^2\right)$$

$$\bar\omega_+\bar\omega_- \underbrace{-\frac{u_0'}{2}\bar\omega_+ +\frac{u_0'}{2}\bar\omega_-}_{+2\frac{u_0'}{2}ku_0} \cong +\left(\frac{u_0'}{2}\right)^2 4ka - \left(\frac{u_0'}{2}ka\right)^2 8$$

$$\Rightarrow \bar\omega_+\bar\omega_- \cong -2k^2u_0^2$$

$$\Rightarrow \boxed{\omega^2 = -k^2u_0^2, \quad \lim ka \to 0}$$

$\omega = \pm i k u_0$, as from sharp bndry

General ka

Let $\omega / u_0' \equiv \Omega$

$$\Rightarrow \frac{\bar{\omega}_{\mp}}{u_0'} = \frac{\omega \mp k u_0}{u_0'} = \Omega \mp ka$$

$$\Rightarrow \left(1 + \frac{1}{2\bar{\Omega}_+}\right)\left(1 - \frac{1}{2\bar{\Omega}_-}\right) = -\frac{1}{4}\frac{e^{4ka}}{\bar{\Omega}_+\bar{\Omega}_-}$$

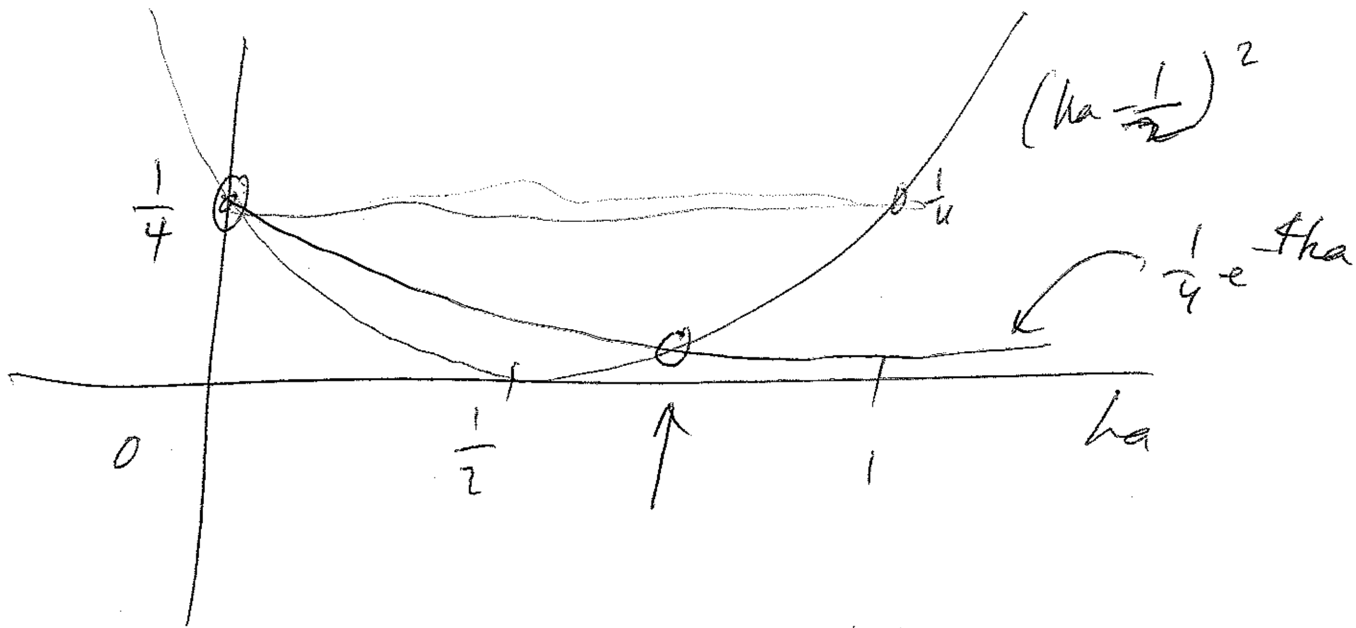
$$\Rightarrow \left(\bar{\Omega}_+ + \frac{1}{2}\right)\left(\bar{\Omega}_- - \frac{1}{2}\right) = -\frac{1}{4}e^{-4ka}$$

$$\Rightarrow \left(\Omega - ka + \frac{1}{2}\right)\left(\Omega + ka - \frac{1}{2}\right) = -\frac{1}{4}e^{-4ka}$$

$$\Rightarrow \quad \Omega^2 - \left(ka - \frac{1}{2}\right)^2 = -\frac{1}{4}e^{-4ka}$$

$$\Rightarrow \boxed{\Omega^2 = \left(ka - \frac{1}{2}\right)^2 - \frac{1}{4}e^{-4ka}}$$

Ω^2 = real, $\therefore$ $\Omega^2 = 0$ gives marginal stability. $\Omega^2 = 0 \Rightarrow \frac{1}{4}e^{-4ka} = \left(ka - \frac{1}{2}\right)^2$



2 roots: $ka = 0$ & $\frac{1}{2} < ka < 1$