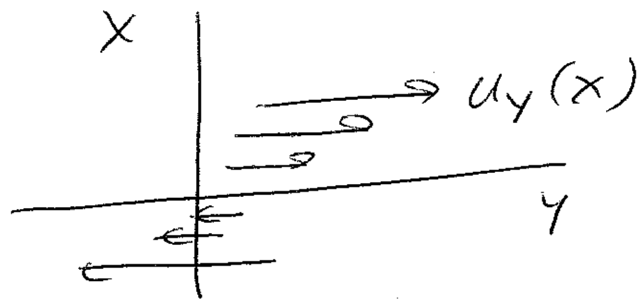
Kelvin Helmholtz Instab from Reduced Eqns

$$\boxed{\frac{u}{L} \ll \frac{c_s}{L}}$$

$$\partial_t n + \vec{\nabla}\cdot(n\vec{u}) = 0$$

$$\vec{\nabla}\cdot\vec{u} \simeq 0 \qquad (g=0)$$

$$\vec{\nabla}\times(n\, d\vec{u}/dt) \simeq 0$$



Equilibrium

$u_y(x)$, $\partial_y = 0$

$n = \text{const}$

$$\Rightarrow \quad \vec{\nabla}\times(\vec{u}\cdot\vec{\nabla}\vec{u}) = 0 \quad \Rightarrow \quad \vec{\nabla}\times\hat{y}\, u_y \partial_y u_y = 0$$

OK $\forall\, u_y(x)$

Perturb

Let $k_z = 0$ (simplicity)

$$\Rightarrow \quad \vec{\nabla}\cdot\tilde{\vec{u}} = 0 \quad \Rightarrow \quad \boxed{\tilde{\vec{u}} = \hat{z}\times\vec{\nabla}\tilde{\varphi}}$$

$$\partial_t \tilde{n} + n\,\vec{\nabla}\cdot\tilde{\vec{u}} = 0 \quad \Rightarrow \quad \boxed{\tilde{n} = 0}$$

$$\therefore \left(\vec{\nabla}\times\frac{d}{dt}\tilde{\vec{u}}\right) = 0, \quad \vec{u} = \hat{z}\times\vec{\nabla}\varphi$$

OK in general

$$\vec{\nabla} \times \left(\frac{d}{dt} \hat{z} \times \vec{\nabla} \varphi \right) = 0$$

$$\Rightarrow \vec{\nabla} \times \left(\hat{z} \times \frac{d}{dt} \vec{\nabla} \varphi \right) = 0 \qquad n_0 = \text{const}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \left(\frac{d}{dt} \vec{\nabla} \varphi \right) = 0} \qquad \varphi = \varphi(x) + \tilde{\varphi}$$

NL $\qquad \tilde{\varphi} = \tilde{\varphi}(x) e^{iky - i\omega t}$

<u>Linearize</u>

$$\vec{\nabla} \cdot \left(\partial_t \vec{\nabla} \tilde{\varphi} \right) + \vec{\nabla} \cdot \left[\vec{u} \cdot \vec{\nabla} \vec{\nabla} \tilde{\varphi} + \tilde{\vec{u}} \cdot \vec{\nabla} \vec{\nabla} \varphi \right] = 0$$

$$\partial_t \nabla^2 \tilde{\varphi} + \vec{\nabla} \cdot \left[u \, \partial_y \vec{\nabla} \tilde{\varphi} \right] + \vec{\nabla} \cdot \left[\hat{x} \tilde{u}_x \varphi'' \right] = 0$$

$$\partial_y \to ik, \quad \partial_t \to -i\omega, \quad \tilde{u}_x = -ik \tilde{\varphi}$$

$$\Rightarrow -\omega \nabla^2 \tilde{\varphi} + k \vec{\nabla} \cdot \left[u \vec{\nabla} \tilde{\varphi} \right] - k \vec{\nabla} \cdot \left[\hat{x} \tilde{\varphi} \varphi'' \right] = 0$$

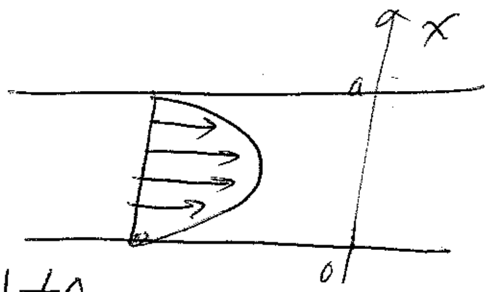
$$u = u_y(x) = \varphi'$$

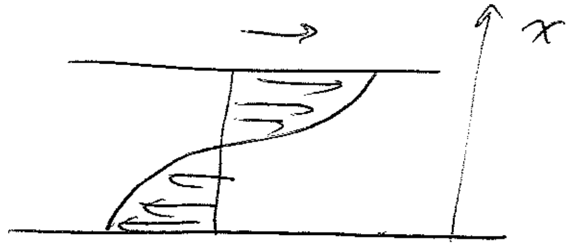
$$-\omega \nabla^2 \tilde{\varphi} + k u \nabla^2 \tilde{\varphi} + \cancel{k u' \tilde{\varphi}'} - \cancel{k \tilde{\varphi}' \varphi''} - k \tilde{\varphi} \varphi''' = 0$$

$$\boxed{(\omega - k u) \nabla^2 \tilde{\varphi} + k u'' \tilde{\varphi} = 0} \qquad \text{eV eqn}$$

$$\boxed{\nabla^2 \equiv \partial_x^2 - k^2} \qquad u(x)$$

Rayleigh-Inflexion Thm





$u'' \neq 0$ $\forall x$ $\qquad u_1(x)$ $\qquad \overleftarrow{u_2}(x)$ $\qquad u''(0)=0$

let $\tilde{\varphi}(0)=0$, $\tilde{\varphi}(a)=0$ (hard walls)

$$\tilde{\varphi}'' - k^2\tilde{\varphi} = -\frac{ku''}{\omega - ku}\tilde{\varphi}$$

$$\int_0^a dx\, \tilde{\varphi}^* \Rightarrow \langle \tilde{\varphi}^* \tilde{\varphi}'' \rangle - k^2 \langle |\tilde{\varphi}|^2 \rangle = -k\left\langle \frac{u''|\tilde{\varphi}|^2}{(\omega - ku)} \right\rangle$$

$$\langle \tilde{\varphi}^* \tilde{\varphi}'' \rangle = \underbrace{\left[\tilde{\varphi}^* \tilde{\varphi}'\right]_0^a}_{=0} - \langle |\tilde{\varphi}'|^2 \rangle$$

$$\Rightarrow \langle |\tilde{\varphi}'|^2 \rangle + k^2 \langle |\tilde{\varphi}|^2 \rangle = k\left\langle \frac{u''|\tilde{\varphi}|^2}{\omega - ku} \right\rangle$$

$$\text{RHS} = k\left\langle \frac{u''|\tilde{\varphi}|^2(\omega^* - ku)}{|\omega - ku|^2} \right\rangle$$

Im (both sides) $\Rightarrow$ $0 = \left\langle \frac{u''|\tilde{\varphi}|^2}{|\omega - ku|^2} \right\rangle \mathrm{Im}(\omega^*)$

$\omega = \omega_r + i\omega_i$

$\mathrm{Im}(\omega^*) = -\omega_i$ $\qquad \boxed{u'' \neq 0 \Rightarrow \omega_i = 0 \Rightarrow \text{stable}}$

INFLEXION Thm