Cold Fluid Eqns

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Moments $\Rightarrow$

$$\partial n_\alpha/\partial t + \vec{\nabla}\cdot(n_\alpha \vec{u}_\alpha) = 0$$

$$\partial_t(n_\alpha m_\alpha \vec{u}_\alpha) + \vec{\nabla}\cdot(n_\alpha m_\alpha \vec{u}_\alpha \vec{u}_\alpha) = -\vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}_\alpha$$
$$+ n_\alpha e_\alpha(\vec{E} + \vec{u}_\alpha \times \vec{B}),$$

etc.

Suppose $\boxed{\omega \gg k v_{th\alpha}}$ (cold response)

then, $\omega \sim ku$

$$\Rightarrow \partial_t nmu : \nabla\cdot\mathbb{P}$$
$$\sim \omega^2 nm : k^2 nT$$
$$\sim \omega^2 : k^2 v_{th}^2$$

$\uparrow$ small

$\therefore$ Neglect $\vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}$ term

$\Rightarrow$ $\boxed{\text{CLOSED, Cold Fluid eqns}}$

/over

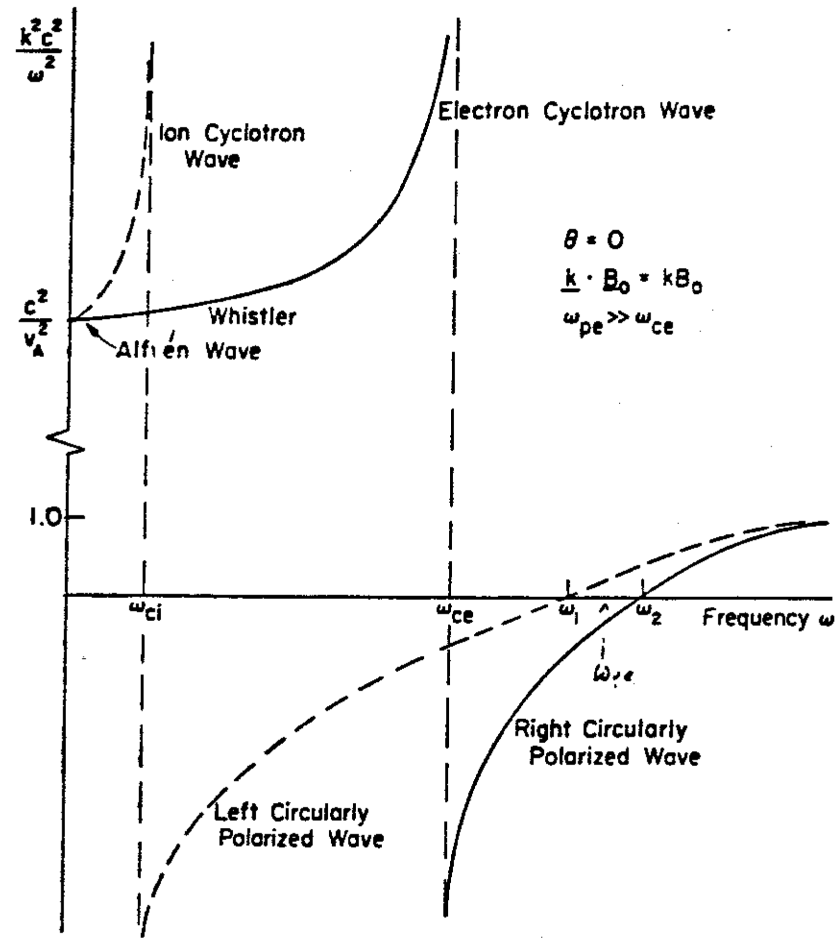


FIGURE 4.10.8a
Index of refraction k^2c^2/ω^2 versus ω for waves that propagate along a magnetic field in a high-density plasma. (Single underscore signifies vector.)

Fig. 1

[Figs. 1, 2, and 3 reproduced from Krall and Trivelpiece, Principles of Plasma Physics.]

Test Problem for Magnetized Plasma

- Braginskii eqns
- Chew-Goldberger-Low Eqns

Collisional MHD Eqns (2D)

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = 0$$

$$nM\frac{d\vec{u}}{dt} = -\vec{\nabla}(p + B^2/2)$$

$$\frac{\partial B}{\partial t} + \vec{\nabla}\cdot(Bu) = 0$$

$$\frac{\partial p}{\partial t} + \vec{u}\cdot\vec{\nabla}p + \gamma p\vec{\nabla}\cdot\vec{u} = 0$$

Linearize

$$\partial_t \tilde{n} + \cancel{n}\vec{\nabla}\cdot\tilde{\vec{u}} = 0$$

$$nM\partial_t\tilde{\vec{u}} = -\vec{\nabla}(\tilde{p} + B\tilde{B})$$

$$\partial_t\tilde{B} + B\vec{\nabla}\cdot\tilde{\vec{u}} = 0$$

$$\partial_t\tilde{p} + \gamma p\vec{\nabla}\cdot\tilde{\vec{u}} = 0$$

$$\Rightarrow \quad nM\omega^2 = k^2\gamma p + k^2 B^2$$

$$\Rightarrow \quad \boxed{\omega^2 = k^2\left[\underbrace{C_s^2}_{\gamma\frac{T}{m}} + \underbrace{V_A^2}_{\frac{B^2}{nM}}\right]}$$

Chew - Goldberger - Low

Eqns

Chew-Goldberger-Low Eqns (Needs checking)

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}\right] \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Scale:

$$\omega \;:\; \frac{v_{th}}{L} \;:\; \frac{eE}{mv_{th}} \;:\; \omega_c$$

let $E \sim v \times B$, MHD ordering

$$\Rightarrow \quad \frac{\omega}{\omega_c} \;:\; \frac{\rho}{L} \;:\; 1 \;:\; 1$$

Let $\frac{\rho}{L} \ll 1$, $\omega \sim k v_{th}$

$$\Rightarrow \quad \epsilon : \epsilon : 1 : 1$$

Let $\vec{v} = \vec{u} + \vec{w}$

$\vec{u}$: fluid speed

$$\Rightarrow \left[\vec{E} + \vec{u} \times \vec{B} + \vec{w} \times \vec{B}\right] \cdot \frac{\partial f_0}{\partial \vec{v}} = 0$$

Let $\vec{u}$ be defined so that

$$\vec{E} + \vec{u} \times \vec{B} \equiv 0$$

$$\Rightarrow \quad \vec{w} \times \vec{B} \cdot \frac{\partial f_0}{\partial \vec{v}} = 0$$

$$f = f(\vec{x}, \vec{v}, t) \rightarrow f(\vec{x}, \vec{w}, t)$$

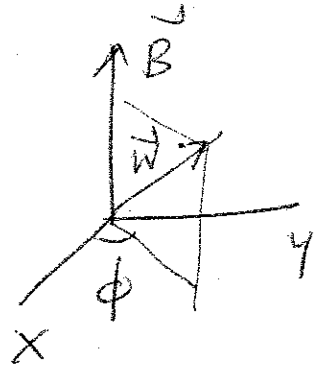
$$\Rightarrow \quad \frac{\partial f}{\partial \vec{v}} = \frac{\partial \vec{w}}{\partial \vec{v}} \cdot \frac{\partial f}{\partial \vec{w}} = \frac{\partial f}{\partial \vec{w}}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial \vec{w}}{\partial t} \cdot \frac{\partial f}{\partial \vec{w}} = \frac{\partial f}{\partial t} \cancel{+} \frac{\partial \vec{u}}{\partial t} \cdot \frac{\partial f}{\partial \vec{w}}$$

and $$\frac{\partial f}{\partial \vec{x}} = \frac{\partial f}{\partial \vec{x}} + \frac{\partial \vec{w}}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{w}} = \frac{\partial f}{\partial \vec{x}} - (\vec{\nabla} \vec{u}) \cdot \frac{\partial f}{\partial \vec{w}}$$

since $\vec{w} = \vec{v} - \vec{u}(\vec{x}, t)$

Then, $$\vec{w} \times \vec{B} \cdot \frac{\partial f_0}{\partial \vec{w}} = 0$$



$$\Rightarrow B \frac{\partial f_0}{\partial \phi} = 0$$

$$\Rightarrow f_0 = f_0(\vec{x}, w_\perp, w_\parallel, t)$$

1st order for $f_1(\vec{x},\vec{w},t)$

using transformations

$$\frac{\partial f_0}{\partial t} + \vec{u}\cdot\vec{\nabla} f_0 + \vec{w}\cdot\vec{\nabla} f_0 - \frac{\partial \vec{u}}{\partial t}\cdot\frac{\partial f_0}{\partial \vec{w}} - (\vec{u}+\vec{w})\cdot(\vec{\nabla}\vec{u})\cdot\frac{\partial f_0}{\partial \vec{w}} + \omega_c \frac{\partial f_1}{\partial \phi} = 0$$

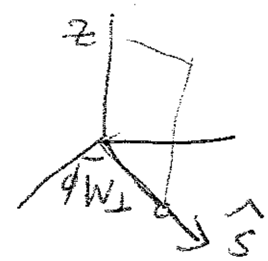
$$\Rightarrow \boxed{\frac{df_0}{dt} + \vec{w}\cdot\vec{\nabla} f_0 - \frac{d\vec{u}}{dt}\cdot\frac{\partial f_0}{\partial \vec{w}} - \vec{w}\cdot(\vec{\nabla}\vec{u})\cdot\frac{\partial f_0}{\partial \vec{w}} + \omega_c \frac{\partial f_1}{\partial \phi} = 0}$$

Annihilate $\oint \frac{d\phi}{2\pi}\frac{\partial f_1}{\partial \phi} = 0$

Apply $\oint \frac{d\phi}{2\pi}$ to above eqn $\Rightarrow$

$$\frac{df_0}{dt} - (\vec{\nabla}\vec{u}) : \overline{\left(\frac{\partial f_0}{\partial \vec{w}}\,\vec{w}\right)} = 0$$

$\vec{w} = \hat{z} w_z + \hat{s} w_\perp$, $\quad \frac{\partial f_0}{\partial \vec{w}} \longrightarrow \hat{s}\frac{\partial f_0}{\partial w_\perp}$



$$\Rightarrow \cancel{then}\ \overline{\frac{\partial f_0}{\partial \vec{w}}\vec{w}} \rightarrow \overline{\frac{\partial f_0}{\partial w_\perp}\hat{s}(\hat{s}w_\perp + \hat{z}w_z)}$$

$$= \frac{\partial f_0}{\partial w_\perp} w_\perp \overline{\hat{s}\hat{s}} = w_\perp \frac{\partial f_0}{\partial w_\perp}(\vec{\vec{I}} - \hat{z}\hat{z})$$

Then, $\vec{\nabla}\vec{u} : \overline{\left(\frac{\partial f_0}{\partial \vec{w}}\vec{w}\right)} \rightarrow w_\perp \frac{\partial f_0}{\partial w_\perp}(\vec{\nabla}_\perp \cdot \vec{u})$

$$\Rightarrow \boxed{\frac{df_0}{dt} - \vec{\nabla}\cdot\vec{u}\ w_\perp \frac{\partial f_0}{\partial w_\perp} = 0}$$

annihilation

Now apply moment equations

Note $\vec{\vec{P}} = \int m\vec{w}\vec{w} f\, d\vec{w}$, definition

$$\Rightarrow \vec{\vec{P}} \rightarrow \int m(w_\perp \hat{s} + w_z \hat{z})^2 f_0\, d\vec{w}$$

$$= \int m f_0\, w_\perp dw_\perp\, dw_z\, \phi\, d\phi \left[w_\perp^2 \hat{s}\hat{s} + w_z^2 \hat{z}\hat{z}\right]$$

$$\Rightarrow \quad \overleftrightarrow{P} \longrightarrow P_\perp\left(\overleftrightarrow{I} - \vec{z}\,\vec{z}\right) + P_\parallel \vec{z}\,\vec{z}$$

$$= \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix}, \quad \text{off-diags} = 0$$

on account of $\dfrac{\partial f_0}{\partial \phi} = 0$

Furthermore, $P_\perp$ moment simplifies $\Rightarrow$ Final CGL system

$$\boxed{\begin{aligned} &\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = 0 \\ &nM\frac{d\vec{u}}{dt} = -\vec{\nabla}\left(P_\perp + \frac{B^2}{2}\right) \\ &\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0 \\ &\frac{\partial B}{\partial t} + \vec{\nabla}\cdot(\vec{u}B) = 0 \end{aligned}}$$

closed $\{n, P_\perp, \vec{u}, B\}$

Linearized CGL & Magson waves

$$\partial_t \tilde{n} + n \vec{\nabla} \cdot \vec{\tilde{u}} = 0$$

$$nM \partial_t \vec{\tilde{u}} = -\vec{\nabla}(\tilde{P}_\perp + B\tilde{B})$$

$$\partial_t \tilde{B} + B \vec{\nabla} \cdot \vec{\tilde{u}} = 0$$

$$\partial_t \left(\frac{\tilde{P}_\perp}{P_\perp} - \frac{\tilde{n}}{n} - \frac{\tilde{B}}{B} \right) = 0$$

$$\Rightarrow \quad nM\omega^2 = k_\perp^2 \left[2P_\perp + B^2 \right]$$

$$\Rightarrow \quad \boxed{\omega^2 = k_\perp^2 \left[2\frac{P_\perp}{nM} + V_A^2 \right]}$$

contrast

$$\gamma \frac{P}{nM}$$