The Equations of Plasma Physics

A. B. Hassam
UMD Tutorial

DO NOT
TRUST my units!

only the structure
of the equations.

Some algebra
may be off!

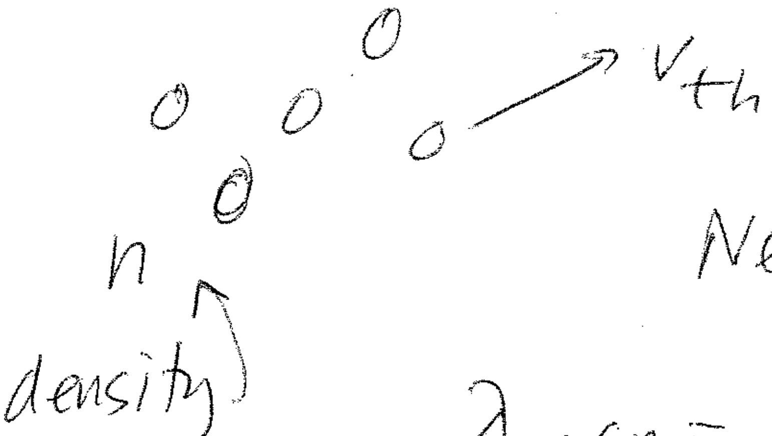
The idea is to
be schematic and
emphasise ~~Eq~~ lation structure.

Derive the algebra
yourself!

Test Problem for
Neutral Gas

⟹ ∘ Moment Eqns
∘ Ideal Fluid Eqns
∘ Navier–Stokes Eqns
∘ Boltzmann Eqn
∘ Vlasov Eqn
∘ Reduced Eqns

Test Problem for Neutral Gases



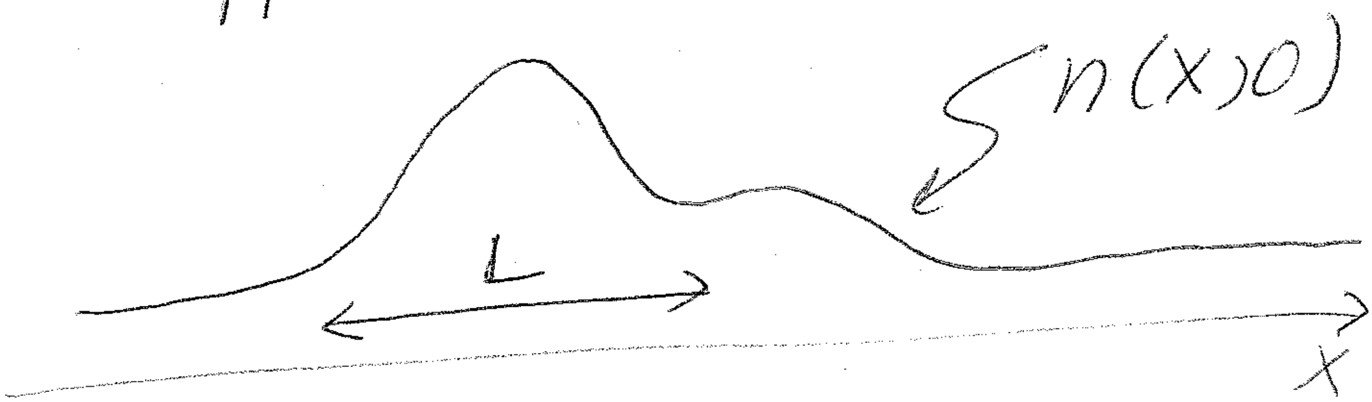
Neutral gas

$$\lambda_{mfp} = \lambda = \frac{1}{n\sigma}$$

@ STP, $\lambda \sim 10^{-4}$ cm

Problem

Suppose at $t=0$,

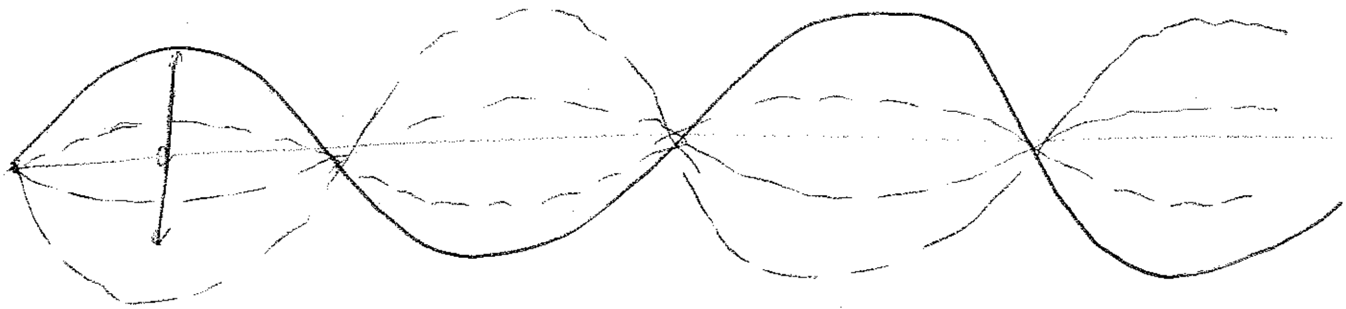


Find $n(x,t)$.

2 cases: $\lambda \ll L$ and $L \ll \lambda$.

Answers to test problem

$\lambda \ll L$



$L \ll \lambda$

First Principles Eqn to Solve test problem

$$\boxed{\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = C(f,f)}$$

Boltzmann eqn

Scaling:

$$\omega \;:\; \frac{v_{th}}{L} \;:\; \nu$$

$$\Rightarrow \underbrace{\frac{\omega}{\nu} \;:\; \frac{\lambda}{L}}_{\substack{\omega \sim \frac{v_{th}}{L} \\ \text{comparable}}} \;:\; \underbrace{1}_{\text{disparate}}$$

Moment Equations

(Neutral Gas only)

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f = C(f)$$

$$\vec{v}\cdot\vec{\nabla} f = \vec{\nabla}\cdot(\vec{v} f)$$

let

$$n \equiv \int d\vec{v}\, f$$

$$n\vec{u} \equiv \int d\vec{v}\, \vec{v} f$$

$$\overleftrightarrow{P} \equiv \int d\vec{v}\, m(\vec{v} - \vec{u})(\vec{v}-\vec{u}) f$$

$$\vec{q} \equiv \int d\vec{v}\, \tfrac{1}{2} m |\vec{v}-\vec{u}|^2 (\vec{v}-\vec{u}) f$$

Moments $\Rightarrow$

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = 0$$

$$\frac{\partial}{\partial t}(nM\vec{u}) + \vec{\nabla}\cdot(nM\vec{u}\vec{u}) = -\vec{\nabla}\cdot\overleftrightarrow{P}$$

and, e.g.,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}nMu^2 + \frac{3}{2}p\right) + \vec{\nabla}\cdot\left(\left[\frac{nMu^2}{2} + \frac{3}{2}p\right]\vec{u}\right) + \vec{\nabla}\cdot\overleftrightarrow{P}\cdot\vec{u} = -\vec{\nabla}\cdot\vec{q}$$

NOT closed.

Chapman-Enskog Expansion

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f = C(f) \quad — (1)$$

$$f(\vec{x}, \vec{v}, t)$$

Sizes of (1) $\Rightarrow$

$$\omega \; : \; \frac{v_{th}}{L} \; : \; \nu$$

$$\sim \quad \frac{\omega}{\nu} \; : \; \frac{\lambda}{L} \; : \; 1$$

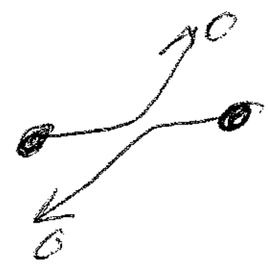
let $\lambda \ll L$, let $\omega \sim v_{th}/L$

$$\Rightarrow \quad \epsilon \; : \; \epsilon \; : \; 1$$

$\therefore$ $C(f)$ is large operator term

Properties of Collision Operator

$$C(f,f) = C(f) \sim \underset{\substack{\uparrow \\ \text{collision frequency}}}{\nu} f$$

Randomizing Collisions 

Conserve number, momentum, and KE (for weak forces)

$$\Rightarrow \boxed{\begin{aligned} &\int d\vec{v}\, C(f) = 0 \\ &\int d\vec{v}\, m\vec{v}\, C(f) = 0 \\ &\int d\vec{v}\, \tfrac{1}{2} m v^2\, C(f) = 0 \end{aligned}}$$

H Theorem: $C(f) = 0$

$$\Rightarrow f = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{\frac{1}{2}m|\vec{v}-\vec{u}|^2}{T}}$$

Annihilator

- Suppose $Lf = g$, g is given (operator: L)
  find f (inhomogeneous eqn)

- Theorem f exists only if g satisfies certain conditions.

Proof Suppose $\exists$ "annihilator" operator $A \ni AL = 0$

Then, $Lf = g \Rightarrow ALf = Ag$

$$\Rightarrow \boxed{0 = Ag}$$ QED

$\therefore$ $\boxed{f \text{ exists only if } Ag = 0 \;\; \forall A \text{ where } AL = 0}$ g cannot be arbitrary

Expand in $C \to \infty$

$$\Rightarrow \quad 0 = C(f_0)$$

$$\Rightarrow \quad f_0 = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{1}{2}\frac{m|\vec{v}-\vec{u}|^2}{T}}$$

where $n = n(\vec{x},t)$, $T = T(\vec{x},t)$
$\vec{u} = \vec{u}(\vec{x},t)$

$\therefore$ For any point in space-time, f = maxwellian, though temperatures, etc, may vary in spacetime.

1st order

$$\frac{\partial f_0}{\partial t} + \vec{v}\cdot\vec{\nabla} f_0 = C(f_1) \quad \text{—} (2)$$

inhomogeneous eqn for f_1

Solutions f_1 exist provided annihilators operate on inhomogeneity

From properties of $C(f)$

$$\int d\vec{v} \begin{Bmatrix} 1 \\ m\vec{v} \\ \frac{1}{2}mv^2 \end{Bmatrix} C(f_1) = 0$$

$$\Rightarrow \int d\vec{v} \begin{Bmatrix} 1 \\ m\vec{v} \\ \frac{1}{2}mv^2 \end{Bmatrix} \left[\frac{\partial f_0}{\partial t} + \vec{v} \cdot \frac{\partial f_0}{\partial \vec{x}} \right] = 0$$

3 moments, but with $f \to f_{max}$

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0 \\ &nM\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u}\right) = -\vec{\nabla} p \\ &\frac{\partial p}{\partial t} + \vec{u} \cdot \vec{\nabla} p + \gamma p \vec{\nabla} \cdot \vec{u} = 0 \end{aligned}} \quad 3 \quad \gamma = \frac{5}{3}$$

"ideal" fluid eqns