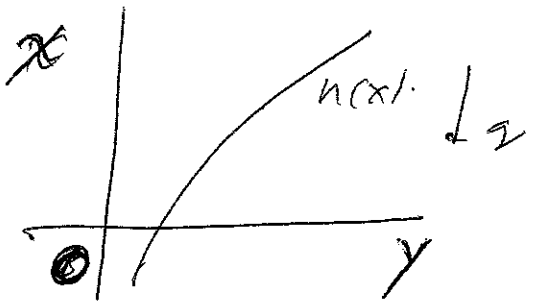
# Soln to #6/702/S07

$$n(x) = n_0 e^{x/L}$$




let $\frac{g}{L} \ll \left(\frac{C_s}{L}\right)^2$

$\Rightarrow$ subsonic

$\Rightarrow$ can use Reduced eqns

assume $\partial_z = 0 \Rightarrow$

$$\hat{z} \cdot \vec{\nabla} \times (nM d_t \vec{u}) = \hat{z} \cdot \vec{\nabla} n \times \vec{g} M$$

$$\partial_t n + \vec{u} \cdot \vec{\nabla} n = 0$$

$$\vec{u} = \hat{z} \times \vec{\nabla} \varphi$$

$e^{ik_y y - i\omega t} \Rightarrow$ eigenvalue eqn as done before

$$\vec{\nabla} \cdot (n \vec{\nabla} \tilde{\varphi}) = \frac{g n'}{\omega^2} k_y^2 \tilde{\varphi}$$

$$\Rightarrow \frac{d^2 \tilde{\varphi}}{dx^2} - k_y^2 \tilde{\varphi} + \frac{n' d\tilde{\varphi}}{n\, dx} = \frac{g n'}{\omega^2 n} k_y^2 \tilde{\varphi}$$

for the profile, $\frac{n'}{n} = \frac{1}{L}$

$$\Rightarrow \tilde{\varphi}'' - k_y^2 \tilde{\varphi} + \frac{\tilde{\varphi}'}{L} = \left(\frac{g/L}{\omega^2}\right) k_y^2 \tilde{\varphi}$$

2nd order ODE, coeffs indep of $x$

Since coeffs indep of $x$, can try

$$\tilde{\varphi}(x) \to \tilde{\varphi} e^{ik_x x}$$

But $\mathrm{Im}(k_x) = 0$ since $\mathrm{Im}(k_x) \neq 0$

$\Rightarrow$ Solns diverge as $|x| \to \infty$ for any sign of $\mathrm{Im}(k_x)$.

$\therefore$ $k_x$ real $\Rightarrow$ $-(k_x^2 + k_y^2) + \frac{2ik_x}{L} = \frac{(g/L)}{\omega^2} k_y^2$

$$\Rightarrow \boxed{\left(\frac{g/L}{\omega^2}\right) = -\left(1 + k_x^2/k_y^2\right) + \frac{2ik_x}{k_y^2 L}}$$

d.r. $\omega = \omega(k_x, k_y)$

(1.) Suppose $k_x \sim k_y$ and $k_y L \gg 1$

then on RHS $1 \gg \frac{k_x}{k_y^2 L} \sim \frac{1}{k_y L}$

$$\Rightarrow \frac{g/L}{\omega^2} \simeq -\left(1 + k_x^2/k_y^2\right) \Rightarrow \boxed{\omega^2 \simeq \frac{-(g/L)}{\left(1 + k_x^2/k_y^2\right)}}$$

Correction $-\frac{(g/L)}{\omega^3} 2\delta\omega \simeq \frac{ik_x}{k_y^2 L}$

For $L > 0$ $\Rightarrow$ unstable

For $L < 0$ $\Rightarrow$ stable to lowest order, oscillatory

But $\delta\omega \propto -\frac{ik_x}{|L|}\omega$ $\Rightarrow$ weakly unstable

(2) $h_x \sim h_y$ and $h_y L \ll 1$

$\Rightarrow$ on RHS, $ih_x$ term dominates

$$\Rightarrow \frac{gL}{\omega^2} \sim \frac{ih_x}{h_y^2 L} \Rightarrow \boxed{\frac{gh_y}{\omega^2} \sim \frac{ih_x}{h_y}}$$

$$\Rightarrow \omega^2 \propto -i = e^{-i\pi/2} \Rightarrow \omega \propto \pm e^{-i\pi/4}$$

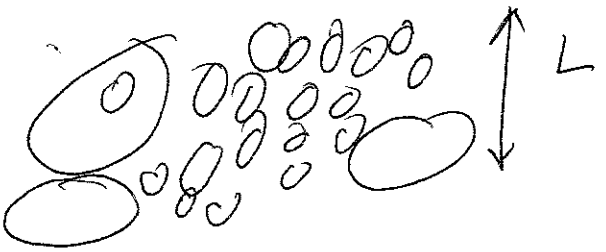
$\therefore$ <u>UNstable</u> with $\omega_r = \omega_i$

indep of sgn(L)

(3) <u>$L > 0$</u> Strongly unstable at short $\lambda$

weaker at long $\lambda$

$\Rightarrow$

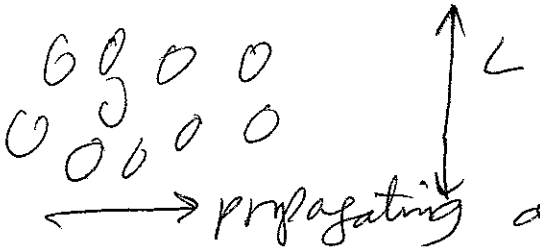



strongly turbulent

<u>$L < 0$</u> weakly unstable at short $\lambda$

similar at long $\lambda$

MYSTERY!
Why unstable for $L < 0$?




$\longrightarrow$ propagating also