# Perturbation Theory – Regular, algebraic eqn

Suppose $\quad x^2 - 2\epsilon x - 1 = 0 \quad$ — (1)

$$\Rightarrow \quad \boxed{x = \epsilon \pm (1+\epsilon^2)^{1/2}} \quad (2) \text{ exact solutions}$$

## Perturbative Solution

let $\quad x = x_0 + \cancel{\epsilon} x_1 + x_2 + \cdots, \quad |x_{n+1}| \ll |x_n|$

Plug into (1) and proceed order by order

$$\Rightarrow \quad x_0^2 - 1 = 0 \quad \Rightarrow \quad x_0 = \pm 1$$

$$2x_0x_1 - 2\epsilon x_0 = 0 \quad \Rightarrow \quad x_1 = \epsilon$$

$$2x_0x_2 + x_1^2 - 2\epsilon x_1 = 0 \quad \Rightarrow \quad x_2 = \pm \frac{1}{2}\epsilon^2$$

$$\Rightarrow \quad \boxed{x = \pm 1 + \epsilon \pm \frac{1}{2}\epsilon^2 + \cdots} \quad (3)$$

in agreement with (2)

$x_1 \ll x_0 \iff \epsilon \ll 1$

$x_2 \ll x_1 \iff \epsilon \ll 1 \qquad$ OK

# Singular Pert Theory (algebraic)

Previously $x^2 - 2\epsilon x - 1 = 0$

Now suppose $\boxed{\epsilon x^2 - x + 1 = 0}$

Naive $\epsilon \to 0 \Rightarrow x = x_0 + x_1 + \cdots$

$\Rightarrow -x_0 + 1 = 0 \Rightarrow x_0 = 1$

$\epsilon x_0^2 - x_1 = 0 \Rightarrow x_1 = \epsilon$

$\underbrace{\epsilon(x_0 + x_1)^2}_{2\epsilon x_0 x_1} - x_2 = 0$

$\Rightarrow x_2 = 2\epsilon^2$

$$\boxed{x = 1 + \epsilon + 2\epsilon^2 + \cdots}$$

ok for $\epsilon \ll 1$

Lost a solution!

assumed $\epsilon x^2 \ll x \Rightarrow \epsilon x \ll 1$

But $x$ could be large? violated if $x \sim \frac{1}{\epsilon}$

So let $x \to \cancel{X} \gg 1$

$\Rightarrow \epsilon X^2 - X + 1 = 0$

since $X \gg 1 \Rightarrow \epsilon X_0^2 - X_0 = 0$

$$\Rightarrow \boxed{X_0 = 1/\epsilon}$$

$$\Rightarrow \underbrace{\epsilon(X_0 + X_1)^2}_{2\epsilon X_0 X_1} - X_1 + 1 = 0$$

$$\Rightarrow 2\epsilon\left(\frac{1}{\epsilon}\right)X_1 - X_1 + 1 = 0$$

$$\Rightarrow 2X_1 + 1 = 0 \Rightarrow \boxed{X_1 = -\frac{1}{2}}$$

etc $\Rightarrow \boxed{X = \frac{1}{\epsilon} + \frac{1}{2} + \ldots O(\epsilon)}$

# Solution of ODE by perturbation (regular) (example)



Suppose $\boxed{y'' - y - \epsilon y^3 = 0}$; we want to find $y(x)$ for $x \geqslant 0$ and $y(x \to \infty) \to 0$, for $\epsilon \ll 1$. Solve perturbatively letting $y = \sum_{n=0}^{\infty} y_n$, demand $|y_{n+1}| \ll |y_n|$.

Match order by order. Then,

$\epsilon^0 \Rightarrow y_0'' = y_0 \Rightarrow \boxed{y_0 = e^{-x}}$, $\rightarrow 0$ at $\infty$ * (ignore constant)

$\epsilon^1 \Rightarrow y_1'' - y_1 = \epsilon \, y_0^3 = \epsilon e^{-3x}$

$y_1 = A_1 e^{-3x} \Rightarrow -8A_1 = \epsilon$

$\Rightarrow \boxed{y_1 = -\frac{\epsilon}{8} e^{-3x}}$ * consider $e^{-x}$ soln only

$$\epsilon^2: \quad y_2'' - y_2 = \underbrace{\epsilon (y_0 + y_1)^3}_{\text{to 2nd order}}$$

$\epsilon (y_0 + y_1)^3 \rightarrow \epsilon 3 y_0^2 y_1$ is 2nd order

$$\Rightarrow y_2'' - y_2 = -\frac{3\epsilon^2}{8} e^{-2x} e^{-3x}$$

$$\Rightarrow \quad y_2'' - y_2 = -\frac{3}{8}\epsilon^2 e^{-5x}$$

Try $y_2 = A e^{-5x} \Rightarrow -24A = -\frac{3}{8}\epsilon^2$

$$A = \epsilon^2/64$$

$$\Rightarrow \quad y_2 = \frac{\epsilon^2}{64} e^{-5x}$$

$$\boxed{y \simeq e^{-x} - \frac{\epsilon}{8} e^{-3x} + \frac{\epsilon^2}{64} e^{-5x} + \cdots}$$

$|y_{n+1}| \ll |y_n|$ ok for $\epsilon \ll e^{2x}$

$$\Leftrightarrow \quad \boxed{\epsilon \ll 1}$$

# ODE Perturbation — singular

Previously $y'' - y - \epsilon y^3 = 0$

Suppose $\boxed{-\epsilon y'' + y' - y = 0}$ $\epsilon \ll 1$

Naive $y = y_0 + y_1 + \cdots ,$

$\Rightarrow \quad y_0' - y_0 = 0 \Rightarrow \boxed{y_0 = e^x}$

$\epsilon y_0'' + y_1' - y_1 = 0$

$\Rightarrow y_1' - y_1 = +\epsilon e^x$

$\Rightarrow$ Try $y_1 = A x e^x , \; y_1' = A e^x + A x e^x$

$A e^x + \cancel{A x e^x} - \cancel{A x e^x} = +\epsilon e^x$

$A = +\epsilon$

$\Rightarrow \boxed{y_1 = \epsilon x e^x}$

$$\boxed{\begin{array}{c} y = e^x + \epsilon x e^x + \cdots \\ \Leftrightarrow \epsilon x \ll 1 ! \end{array}}$$

OK $x \sim 1$

lost a solution!

2nd soln: $y'$ large?

Try $y = e^{S(x)}$, $|S| \gg 1$

$y' = S' e^S$, $y'' = S'^2 e^S + S'' e^S$

$\Rightarrow \epsilon(S'^2 e^S + S'' e^S) + S' e^S - e^S = 0$

$\boxed{-\epsilon S'^2 - \epsilon S'' + S' - 1 = 0}$ S eqn

$S$ lg $\Rightarrow |S'| \gg 1$ small

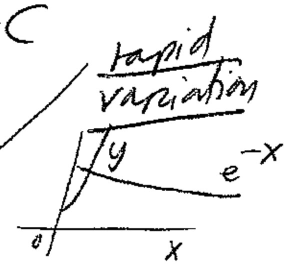
also $S'^2 \gg S''$

$\Rightarrow -\epsilon S'^2 + S'$ left

$\Rightarrow -\epsilon S_0' + 1 = 0 \Rightarrow S_0' = \frac{1}{\epsilon}$

$\Rightarrow \boxed{S_0 = x/\epsilon} + C$

$\Rightarrow \boxed{y_0 = Ce^{x/\epsilon}}$ $y_0 = Ce$




$-\cancel{\epsilon}2\cancel{S_0'}S_1' - \cancel{\epsilon S_0''} + S_1' - 1 = 0$

$-2S_1' + S_1' - 1 = 0$, $S_1' = -1$, $S_1 = -x$

ok for $\epsilon \ll x \sim 1 \Rightarrow \boxed{y = e^{x/\epsilon} + e^{-x}}$ $x \sim 1 \Rightarrow e^{1/\epsilon} \gg e^{-1}$; $x/\epsilon \sim 1 \Rightarrow e^1 \sim 1$