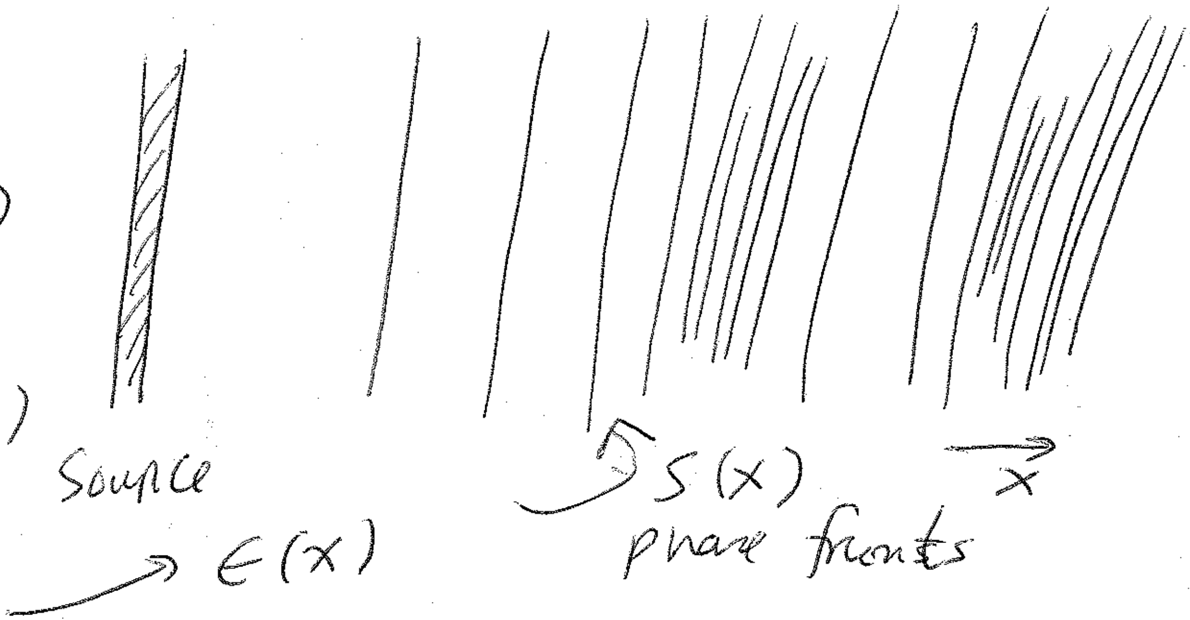
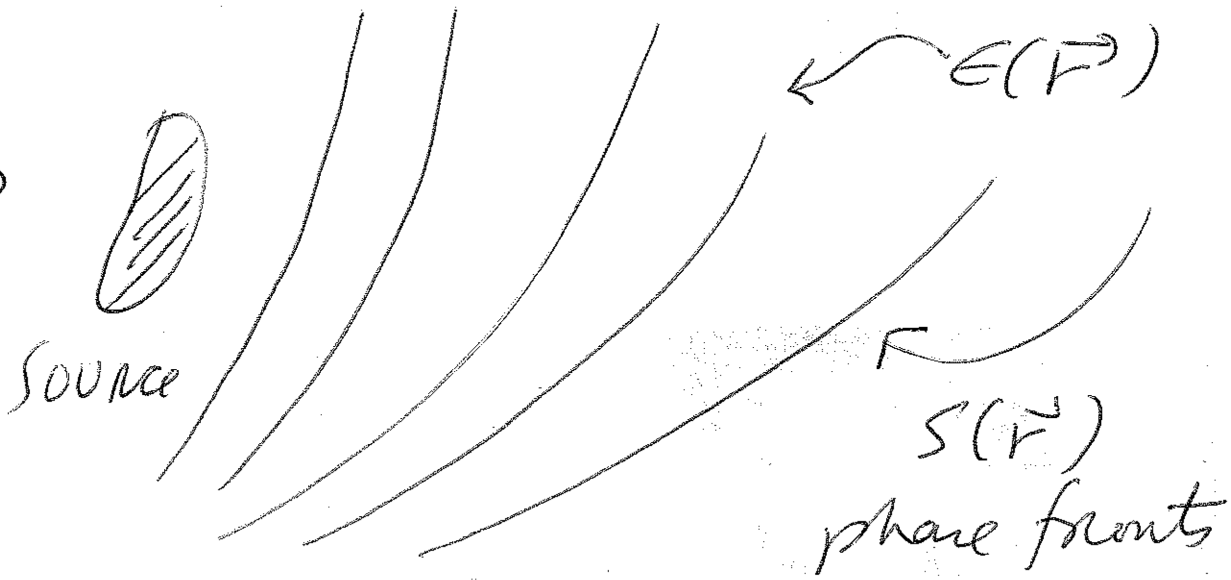
Geometric Optics from WKB in 3D

1-D

$k = S'(x)$



3-D



$\vec{k} = \vec{\nabla} S$

EXAMPLE Consider EM wave in 3D dielectric

So; $\epsilon \frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi \qquad \epsilon(\vec{r})$

$\Rightarrow \quad \nabla^2 \psi = -\omega^2 \psi \epsilon(\vec{r})$

Let $\psi \to e^{iS(\vec{r}) + i\omega t}, \quad S(\vec{r})$

$\Rightarrow \vec{\nabla}\psi = i \vec{\nabla} S e^{iS}$

$\Rightarrow \nabla^2 \psi = -|\vec{\nabla} S|^2 e^{iS} + i \nabla^2 S e^{iS}$

let $\boxed{\vec{k}(\vec{r}) \equiv \vec{\nabla} S}$

$\Rightarrow \boxed{k^2 - i \vec{\nabla} \cdot \vec{k} = \omega^2 \epsilon(\vec{r})}$

exact

For $kL \gg 1$,

$k \to \infty$

$\vec{k} = \vec{k}_0 + \vec{k}_1 + \cdots \quad \Rightarrow \boxed{k_0 = \pm \omega \sqrt{\epsilon}}$

$\Rightarrow \quad |\vec{\nabla} S| = \pm \omega \sqrt{\epsilon}$

$$\Rightarrow \boxed{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = \omega^2 \epsilon(x,y,z)}$$

Hamilton-Jacobi eqn

in general, $\omega = \omega(\vec{k}, \vec{r})$

$S(\vec{r})$ = surfaces of const phase

not easy to solve H-J eqn;

However, just as in 1-D, we can track wave-packets at $\vec{r}(t)$ by using group velocity, as follows:

We have

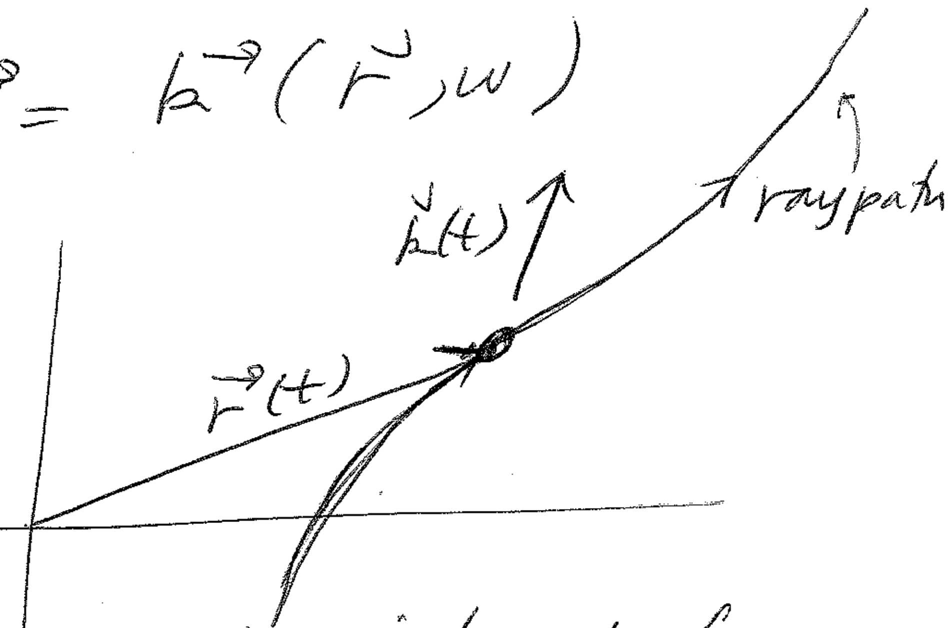
$$\omega = \omega(\vec{k}, \vec{r})$$

$$\Rightarrow \vec{k} = \vec{k}(\vec{r}, \omega)$$

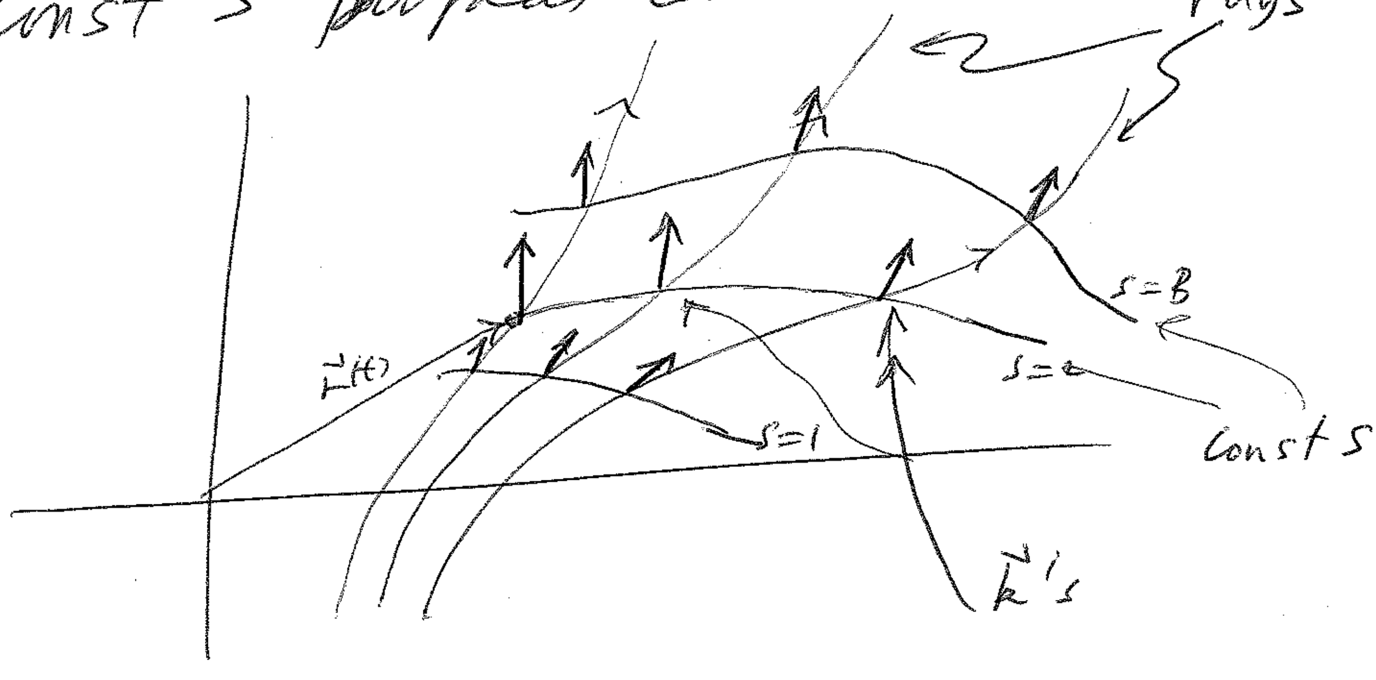
Thus,

if $\vec{r}(t)$

is parametrized,



so that a ray path is found from some initial $\vec{r}(0)$, then there is an assoc $\vec{k}(t)$, with $\vec{k} = \vec{\nabla} S$, const S surfaces can be constructed.



Thus, $\vec{r}(t) \Rightarrow \vec{k}(t)$ parameterization yields $S(\vec{r})$

$$\Rightarrow e^{iS(\vec{r}) - i\omega t}, \quad \vec{k}$$

[Note: $dS(\vec{r}) = \frac{\partial S}{\partial \vec{r}} \cdot d\vec{r}$

$$\Rightarrow dS(\vec{r}) = \vec{k} \cdot d\vec{r}$$

$$\Rightarrow S(\vec{r}) = \int \vec{k} \cdot d\vec{r} \,].$$

Again, as in 1-D, let us choose to parameterize $\vec{r}(t)$ as a "particle" or wavepacket moving with $\vec{V_g}$, $\vec{V_g} = \partial \omega / \partial \vec{k}$

$$\Rightarrow_{\text{let}} \boxed{\dot{\vec{r}} \equiv \partial \omega / \partial \vec{k}} \quad (1)$$

Now $\omega = \omega(\vec{k}, \vec{r})$

then, Method ①*, we note $\frac{d\omega}{dt} = 0$

$$\Rightarrow 0 = \dot{\vec{k}} \cdot \frac{\partial \omega}{\partial \vec{k}} + \dot{\vec{r}} \cdot \frac{\partial \omega}{\partial \vec{r}}$$

from ① $\Rightarrow$ $$0 = \vec{k} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot \frac{\partial \omega}{\partial \vec{r}}$$

$$0 = \left(\dot{\vec{k}} + \partial\omega/\partial\vec{r}\right) \cdot \dot{\vec{r}}$$

$$\Rightarrow \boxed{\dot{\vec{k}} = -\partial\omega/\partial\vec{r}}$$

But, not neccesarily, since we could have $\dot{\vec{k}} + \frac{\partial \omega}{\partial \vec{r}} = \vec{A}$,

where $\vec{A} \cdot \dot{\vec{r}} = 0$.

* faster than Method ②, later

Another method,

Method ②

$$\omega = \omega(\vec{k}, \vec{r})$$

Now $\omega = \text{const} \Rightarrow \dfrac{d\omega}{dx_m} = 0$

$$\Rightarrow 0 = \frac{\partial k_\ell}{\partial x_m}\frac{\partial \omega}{\partial k_\ell} + \frac{\partial \omega}{\partial x_m}$$

Now, $\dot{x}_\ell = \dfrac{\partial \omega}{\partial k_\ell} \Rightarrow 0 = \dfrac{\partial k_\ell}{\partial x_m}\dot{x}_\ell + \dfrac{\partial \omega}{\partial x_m}$

But $k_\ell = \partial s/\partial x_\ell \Rightarrow \dfrac{\partial k_\ell}{\partial x_m} = \dfrac{\partial k_m}{\partial x_\ell}$

$$\therefore \quad 0 = \dot{x}_\ell \frac{\partial k_m}{\partial x_\ell} + \frac{\partial \omega}{\partial x_m}$$

But $k_m(x_\ell) \Rightarrow \dot{k}_m = \dot{x}_\ell \dfrac{\partial k_m}{\partial x_\ell}$

$$\Rightarrow 0 = \dot{k}_m + \partial \omega/\partial x_m$$

$$\Rightarrow \boxed{\dot{\vec{k}} = -\frac{\partial \omega}{\partial \vec{r}}}$$

$$\boxed{\begin{aligned} \dot{\vec{r}} &= \partial \omega / \partial \vec{k} \\ \dot{\vec{k}} &= -\partial \omega / \partial \vec{r} \end{aligned}}$$

$$\left\{ \begin{matrix} \vec{r}(t) \\ \vec{k}(t) \end{matrix} \right\} \quad , \quad \text{Hamiltonian system}$$

indep vars

6 ODE's.

<u>Example</u>

$$+i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \vec{\nabla}^2 \psi}{2m} + V(\vec{r})\psi$$

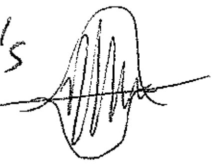
$$\Rightarrow \boxed{\hbar\omega = \frac{\hbar^2}{2m} \underbrace{|\vec{\nabla} S|^2}_{\vec{k}\cdot\vec{k}} + V(\vec{r})}$$

3-D H-J eqn

$$\Rightarrow \hbar \frac{\partial \omega}{\partial \vec{r}} = -\vec{\nabla} V ; \quad \hbar \frac{\partial \omega}{\partial \vec{k}} = \frac{\hbar^2 \vec{k}}{m}$$

$$\Rightarrow \boxed{\begin{aligned} m\dot{\vec{r}} &= \hbar \vec{k} \\ \hbar\dot{\vec{k}} &= -\vec{\nabla} V \end{aligned}}$$

Newton's eqns

consistent with Ehrenfest's Theorem 

In general,

$$\begin{pmatrix} \dot{\vec{r}} \\ \dot{\vec{k}} \end{pmatrix} = \begin{pmatrix} \partial\omega/\partial\vec{k} \\ -\partial\omega/\partial\vec{r} \end{pmatrix}$$

$$\left\{ \begin{array}{l} \dfrac{\partial \omega}{\partial t} = 0 \Rightarrow \omega = \text{const} \\ \text{ignorables, etc.} \end{array} \right.$$

$\Rightarrow$ consts of motion