WKB Method

Suppose $\boxed{\epsilon(x)\,\partial_t^2\psi = c^2\partial_x^2\psi,}$ ①

wave equation. $\epsilon(x)$ is dielectric.

For $\epsilon = 1$, we know $k = \omega/c$.

Suppose we know $\frac{\omega}{c}\left|\frac{\epsilon}{\epsilon'}\right| \gg 1$, $\epsilon' \doteq d\epsilon/dx$

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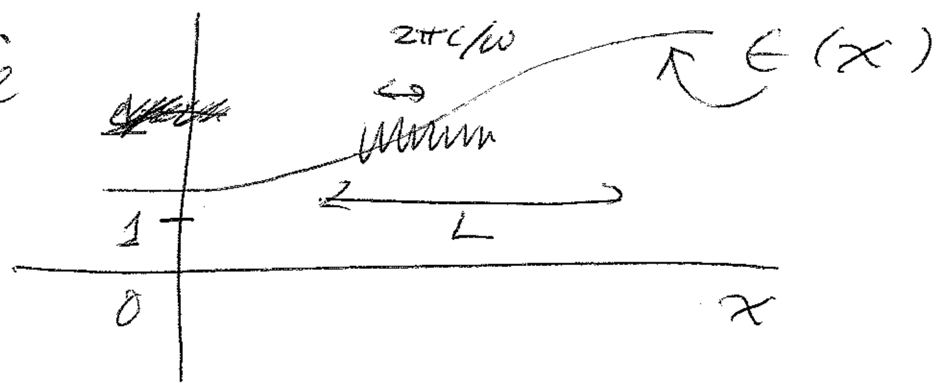


We are expecting wavelength $\ll L$.

Then WKB method applies.

Try $\psi(x,t) = e^{iS(x) - i\omega t}$

expect $S \sim \left(\frac{\omega}{c}\right)x \gg x/L$

Thus, expect $|S|$ to be "large"!

Let us make the ansatz $|S| \gg 1$ and use this as ~~an~~ a perturbation expansion.

Now, $\partial_t^2 \psi \Rightarrow -\omega^2 \psi$

and $\partial_x \psi = i S' \psi$,

$$\partial_x^2 \psi = (-S'^2 + i S'')\psi$$

Then, (1) $\Rightarrow$

$$\boxed{S'^2 - i S'' = \left(\frac{\omega}{c}\right)^2 \epsilon(x)} \quad (2)$$

equation for S' exact

If $S'' \sim S'/L$ and S' "large", then can neglect S'' term, since the ratio of the 2 terms on the LHS $\sim S'^2 : S'/L \sim S'L : 1 \gg 1$.

Let us make this ansatz, to be checked self-consistently upon solution, i.e.

$$\boxed{\text{Ansatz} \quad |S'|^2 \gg |S''|} \quad (3)$$

let $S'(x) \equiv k(x)$.

Then ② $\Rightarrow \quad k^2 - i k' = \left(\frac{\omega}{c}\right)^2 \epsilon(x)$

let $k = \sum_{n=0}^{\infty} k_n$, $\quad |k_{n+1}| \ll |k_n|$

Then, $\quad k_0^2 = (\omega/c)^2 \epsilon(x)$

$$\Rightarrow \quad \boxed{k_0 = \pm (\omega/c) \sqrt{\epsilon(x)}} \quad ④$$

Next, $\quad 2 k_0 k_1 - i k_0' = 0$

$$\Rightarrow \quad \boxed{k_1 = \frac{i}{2} (\ln|k_0|)' + \text{const}} \quad ⑤$$

Demand $|k_1| \ll |k_0|$

(self-consistent check)

$$\Rightarrow \frac{1}{2} \frac{1}{2} \frac{|\epsilon'|}{\sqrt{\epsilon}\sqrt{\epsilon}} \ll \left(\frac{\omega}{c}\right) \sqrt{\epsilon}$$

$$\Leftrightarrow \quad \boxed{\frac{1}{4} \epsilon' \ll \left(\frac{\omega}{c}\right) \epsilon^{3/2}} \quad ⑥$$

condition on validity of WKB ansatz.

$\epsilon \sim 1, \epsilon' \sim \frac{\epsilon}{L} \Rightarrow \quad 1 \ll \left(\frac{\omega L}{c}\right)$. Fails if $\epsilon \to 0$.

Now $S_0 = \int^x dx'\, k_0(x')$

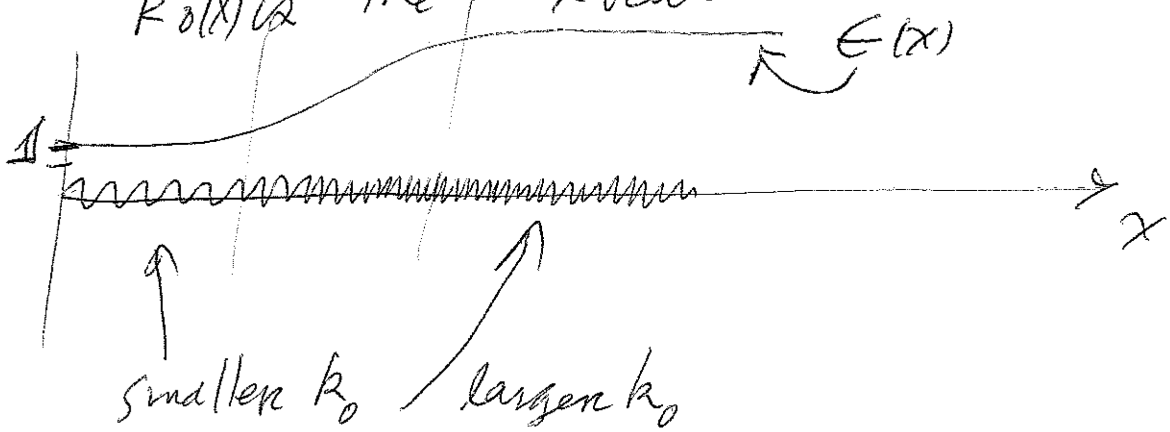
$$S_1 = \int^x dx'\, k_1(x') = \frac{i}{2} \ln |k_0|$$

$$\Rightarrow \boxed{\psi(x,t) \rightarrow C \frac{e^{\pm i \int^x dx' k_0(x') - i\omega t}}{|k_0|^{1/2}}}$$

$$\boxed{k_0(x) = (\omega/c)\sqrt{\epsilon(x)}}$$

Note that $|k_0|^{-1/2}$ comes from 1st order.

$k_0(x)$ is the "local" wavenumber.



if the equivalent $\epsilon(x) \rightarrow 0$, WKB fails as from ⑥.