Hamilton Jacobi Theory for $H = \frac{1}{2}p^2 + V(q)$

- Assume $\boxed{H = \frac{1}{2}p^2 + V(q)}$, $\partial_t H = 0$.

[Somewhat special case but general enough]

- Assume single particle motion in 1-D, for simplicity.
  $\Rightarrow$ 2 initial conditions
  $\Rightarrow$ final $q(t)$ & $p(t)$ will depend on two constants. One of these could be initial energy, for example.

(A) The idea of H-J Theory

Do a CT from $\{p, q\} \to \{P, Q\}$ such that $\{P$ & $Q\}$ are both constants of the motion. One const. is obviously H. So let $P = H$. Then find $Q \Rightarrow \dot{P} = 0$ AND $\dot{Q} = 0$. Lets try this.

(B) Set-up for H-J Theory

- $\{p, q\} \to \{P, Q\} \ni \dot{P} = 0, \dot{Q} = 0$
  and $P = H$

- Since $P = H(p, q)$, try $F_2(q, P, t)$.
  For here, let $F_2 \equiv S$ (standard notation)

  By CT, $\Rightarrow$ $\boxed{\dfrac{\partial S}{\partial q} = p, \; Q = \dfrac{\partial S}{\partial P}}$

  and $K = H + \partial S/\partial t$.
  This implies $\dot{P} = -\partial K/\partial Q$, $\dot{Q} = \partial K/\partial P$

- But, we want $\dot{P} = 0$ and $\dot{Q} = 0$
  $\Rightarrow \partial K/\partial Q = 0$ and $\partial K/\partial P = 0$
  $\Rightarrow$ pick $K = 0$
  $\Rightarrow \boxed{H + \partial S/\partial t = 0}$

- Next, since $\dot{P} = 0 \Rightarrow P = \alpha = \text{const}$
  and $Q = \beta = \text{const}$.

- Our goal now is to find P & Q (or α and β) which will gives $\{p(t), q(t)\}$

(C) Collect all the important equs

- Use the ~~old~~ variables $\{\alpha, \beta, S, q\}$
- Notation is bewildering; so note the re-definitions:

$$P = H = \alpha, \quad Q = \beta,$$

$$p = \pm\sqrt{2[H - V(q)]}, \quad \boxed{\alpha = \frac{p^2}{2} + V(q)}$$

- From the previous boxed equations, using $\{\alpha, \beta, S, q\}$, we have

$\{\alpha, \beta, S, q\}$ system

$$\boxed{\begin{aligned} &① \quad \alpha = \tfrac{1}{2}p^2 + V(q) \quad \leftarrow \text{solves } \alpha(\beta, q) \\ &② \quad \partial S/\partial t = -\alpha \\ &③ \quad \partial S/\partial q = \pm\sqrt{2[\alpha - V(q)]} \\ &④ \quad \beta = \partial S/\partial \alpha \end{aligned}}$$

(D) Solution for $q(t; \alpha, \beta)$, $p(t; \alpha, \beta)$

From ②, $S = W(q, \alpha) - \alpha t$ — ⑤

$\Rightarrow$ From ③, $\partial W/\partial q = \pm\sqrt{2[\alpha - V(q)]}$

$$\Rightarrow W = \pm\int^{q} dq' \sqrt{2[\alpha - V(q')]} \quad — ⑥$$

From (4), using (5)

$\Rightarrow \beta = \partial W/\partial\alpha - t$

$\Rightarrow$, using (6), $\beta + t = \int^{q} dq' \dfrac{1}{\pm\sqrt{2[\alpha - V(q')]}}$ — (7)

(E) <u>We now have a solution as follows</u>

- (7) $\Rightarrow \beta(\alpha, q, t)$,
This can be turned "inside out" to yield $\boxed{q = q(\alpha, \beta, t)}$ ← soln for $q(t)$

- Insert q into (1) $\Rightarrow$
$\boxed{p(\alpha, \beta, t) = \pm\sqrt{2[\alpha - V(q(\alpha,\beta,t))]}}$ ← soln for $p(t)$

We are done!

(F) <u>To get the $\{\alpha, \beta\}$ coordinate system:</u>

From (1), $\alpha(p, q) = \frac{1}{2}p^2 + V(q)$

From (7), $\beta(p, q, t) = \int^{q} dq' \frac{1}{\sqrt{\ }} - t$,
using (1) to insert for α

JS

Example: H. O. [± signs are not important for this case]

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2, \quad V = \frac{1}{2}q^2$$

Use ① → ④, etc.

$$\Rightarrow \boxed{\alpha = \frac{1}{2}p^2 + \frac{1}{2}q^2} \quad (H1)$$

$$⑦ \Rightarrow \beta + t = \int^{q} dq' \frac{1}{\sqrt{2[\alpha - \frac{1}{2}q'^2]}}$$

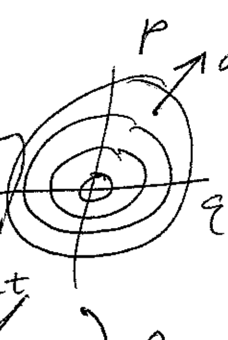
$$= \int^{q/\sqrt{2\alpha}} du \frac{1}{\sqrt{1-u^2}} = \sin^{-1}\left(\frac{q}{\sqrt{2\alpha}}\right)$$

$$\Rightarrow \boxed{q = \sqrt{2\alpha}\,\sin(\beta + t)} \leftarrow q(t) \quad (H2)$$

$$(H2) \rightarrow (H1) \Rightarrow \alpha = \frac{1}{2}p^2 + \alpha \sin^2(\beta + t)$$

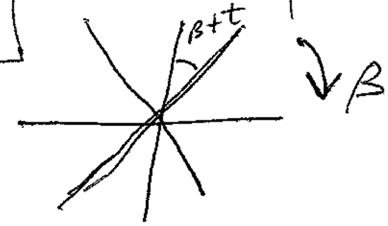
$$\Rightarrow \boxed{p = \sqrt{2\alpha}\,\cos(\beta + t)} \quad (H3)$$

DONE!



To get $\{\alpha, \beta\}$ coordinates: $\boxed{\alpha = \frac{1}{2}p^2 + \frac{1}{2}q^2}$

Divide (H2)/(H3) $\Rightarrow \boxed{\tan(\beta + t) = q/p}$



β rotates as t increases