Find $\{P, Q\}$ coordinates $\ni$ its canonical, Q is periodic, and $P = P(H)$

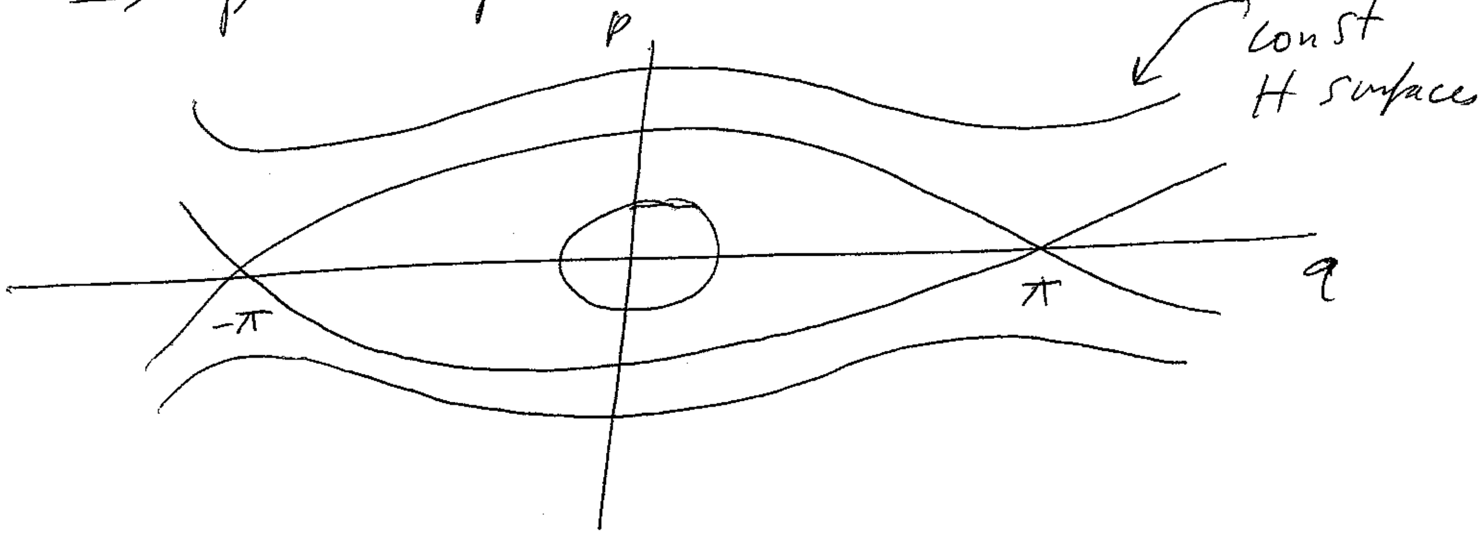
(aka "Action-angle" variables)

- Suppose $\boxed{H = \frac{1}{2}p^2 + V(q)}$

  where $V(q)$ is periodic in $q = [0, 2\pi]$

example, $V(q) = -(\cos q - 1)$

$\Rightarrow$ phase space looks like



- Clearly, we would like to find $P = P(H)$ since this is "natural"!

In particular, for a CT $\ni P = P(H)$

we will have $H = H(P)$ and

$$\dot{P} = -(\partial H/\partial Q) = 0 \Rightarrow \boxed{P = \alpha} = \text{const}$$

and $$\dot{Q} = \frac{\partial H}{\partial P} = H'(P) = H'(\alpha) = \text{const}$$

$$\Rightarrow \boxed{\dot{Q} = \left(\frac{1}{dP/dH}\right)} \Rightarrow Q = \frac{t}{P'(H)} + Q_0$$

easy to integrate and period is $2\pi P'(H)$.

$\boxed{\text{Thus, we want } P(H)}$

- But, is it $P = H$?, $P = \sqrt{H}$?, $P = e^H$, etc. ~~the~~ (provided Q is canonical & periodic)

- Start with $P = P[H(p,q)]$

$\Rightarrow$ Use $F_2(q, P)$

$$\Rightarrow p = \frac{\partial F_2}{\partial q}(q, P)$$

$$\Rightarrow \left.\frac{\partial F_2}{\partial q}\right)_P = \pm\sqrt{2[H-V(q)]}$$

$$\Rightarrow F_2 = \pm\int^q dq'\sqrt{2[H-V(q')]}$$

Now $Q = \frac{\partial F_2}{\partial P} \Rightarrow$

$$\boxed{Q = \pm\int^q dq' \frac{\left(\frac{dH}{dP}\right)}{\sqrt{2[H-V(q')]}}} \quad -(1)$$

- So now we have Q in terms of $P(H)$. Many $P(H)$'s may still work. But we must demand periodicity. We have $Q = Q(q, H)$. For any H, let $Q(0,H)=0$. This fixes the lower bound in (1). Now, outside the separatrix, we let $Q(\pi, H) = \pi$, $\forall H$. This means

$$\int_{-\pi}^{\pi} dq' \frac{1}{\sqrt{2[H-V(q')]}} = 2\pi\frac{dP}{dH}$$

But $\frac{d}{dH}\int_{-\pi}^{\pi} dq\,\sqrt{2[H-V(q)]} = \int_{-\pi}^{\pi} dq\,\frac{1}{\sqrt{2[H-V]}}$

$$\Rightarrow \quad 2\pi \frac{dP}{dH} = \frac{d}{dH}\int_{-\pi}^{\pi} dq\,\sqrt{2(H-V)}$$

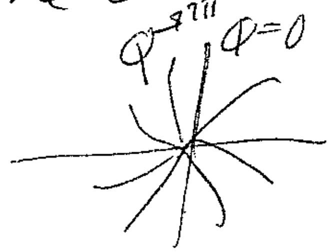
$$\Rightarrow \quad \boxed{P(H) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p\,dq}$$

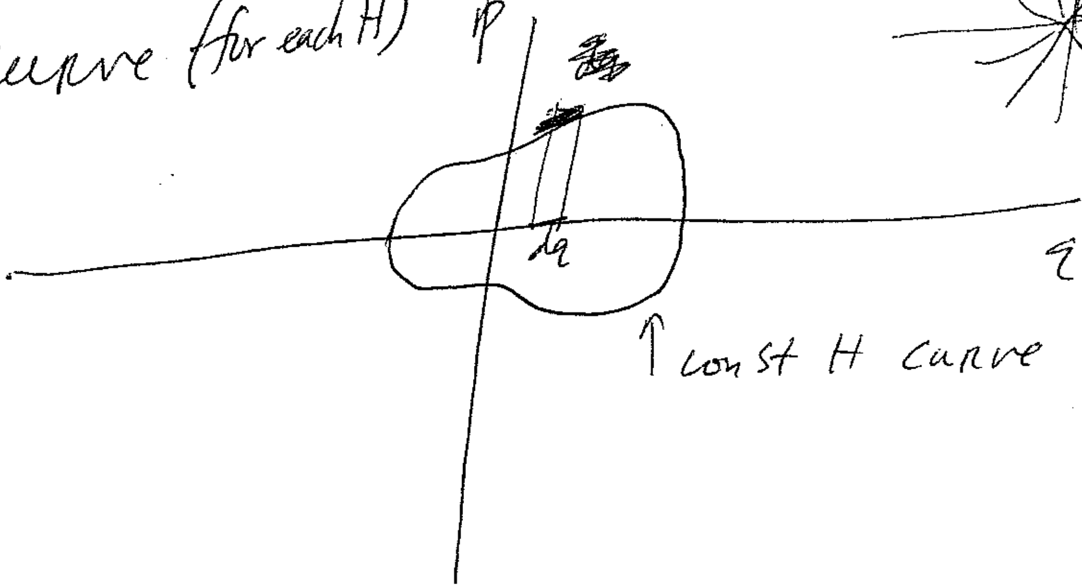
its Area enclosed/2π

outside separatrix

where $p = \pm\sqrt{2(H-V)}$

Inside the separatrix, Q must change by 2π if the $\int dq$ integral at constant H is taken once around the closed curve (for each H)





So, $$2\pi \frac{dP}{dH} = \oint \frac{dq}{\sqrt{2[H - V(q)]}} \quad \text{— (2)}$$

But, if $$P = \frac{\oint dq\, p}{2\pi} = \frac{\text{Area under } H}{2\pi}$$

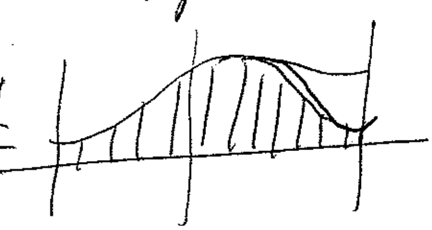
then $$\frac{dP}{dH} = \oint \frac{dq}{2\pi} \frac{\partial}{\partial H} \sqrt{2(H-V)}$$

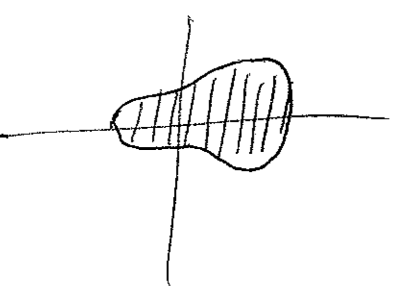
$$= \frac{1}{2\pi} \oint \frac{dq}{\sqrt{2(H-V)}}$$

Compare with (2) $\Longrightarrow$

$$\boxed{P = \frac{1}{2\pi} \oint p\, dq}$$

← Also Area/2π inside separatrix.

Thus, outside, $P = \frac{1}{2\pi}$ 

inside, $P = \frac{1}{2\pi}$ 

Summary (Action-angle variables)

For periodic in q systems,

$H = \frac{1}{2}p^2 + V(q)$, V periodic,

one clearly prefers to transform to variables where $P = P(H)$.

If the transformation is to be canonical and periodic in Q, the only $P(H)$ allowed is

$$\boxed{P(H) = \frac{1}{2\pi}\oint p\,dq}$$

$\oint \to \int_{-\pi}^{\pi}$

$\oint \to$ closed curve

This gives us $P(H)$ explicitly.

Knowing this, $\dot{P} = 0 \Rightarrow P = \alpha = \text{const}$

and $\dot{Q} = \frac{dH}{dP} = \frac{1}{dP/dH} = \text{const}$

$$\Rightarrow \boxed{Q = \frac{t}{P'(H)} + Q_0} \quad \therefore \quad \boxed{\omega = \frac{dH}{dP}}$$

Another Proof of $P(H)$ – Quicker

- Suppose we use $F_1(q, Q)$

$$\Rightarrow \quad p = \partial F_1 / \partial q, \quad P = -\partial F_1 / \partial Q$$

- Now $dF_1 = \dfrac{\partial F_1}{\partial q} dq + \dfrac{\partial F_1}{\partial Q} dQ$

$$\Rightarrow \quad dF_1(q, Q) = p\, dq - P\, dQ$$

- <u>outside</u> the separatrix, integrate dF_1 on constant H curve. Then,

$$\int_{q=-\pi}^{q=\pi} dF_1 = \left[F_1 \right]_{q=-\pi}^{q=\pi} = 0$$

since q & Q are periodic $\angle$'s and F_1 is periodic

$$\Rightarrow \quad \int_{-\pi}^{\pi} p\, dq = P(H)\, 2\pi, \quad \text{since } \int_{-\pi}^{\pi} dQ = 2\pi$$

- <u>inside</u> the separatrix, $\oint dF_1 = 0$

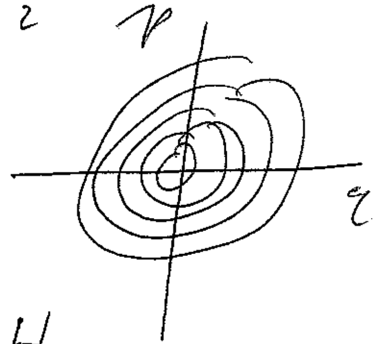
$$\Rightarrow \quad \oint p\, dq = P(H)\, 2\pi, \quad \text{since } \oint dQ = 2\pi$$

$$\Rightarrow \quad \boxed{P(H) = \frac{1}{2\pi} \oint p\, dq}$$

as before

Example

For the H.O., $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$



we want $P = \oint \frac{p\,dq}{2\pi} = \frac{\text{Area}}{2\pi}$

$$\Rightarrow P \to P(p,q) = \frac{\pi(p^2+q^2)}{2\pi} = H$$

$$\therefore \boxed{P = \frac{1}{2}(p^2+q^2) = H}$$

Then, $\dot{Q} = \partial H/\partial P = dH/dP = 1$

$$\Rightarrow \boxed{Q = Q_0 + t}$$

Period of H.O. is given by $\frac{2\pi}{dH/dP} = T$. $\Rightarrow \boxed{T = 2\pi}$