

PS 6 solution

6.1

1)

a)

$$\mathbf{F} = C(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})e^{-\alpha(x^2+y^2)}$$

Check:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \\ &= \left(y(-2\alpha x)e^{-\alpha(x^2+y^2)} - x(-2\alpha y)e^{-\alpha(x^2+y^2)} \right) \hat{\mathbf{z}} \\ &= 0\end{aligned}$$

\therefore potential energy function exists.

$$\begin{aligned}\mathbf{F} &= C(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})e^{-\alpha(x^2+y^2)} \\ &= -\nabla V \\ \implies \frac{\partial V}{\partial x} &= -Ce^{-\alpha(x^2+y^2)}x \\ \implies V &= -\int Ce^{-\alpha x^2} x dx + f(y)\end{aligned}$$

Put $s = x^2 \implies 2x dx = ds$

$$\implies V = -\frac{1}{2} \int Ce^{-\alpha s} ds + f(y)$$

$$\begin{aligned}V &= \frac{Ce^{-\alpha s}}{2\alpha} + f(y) \\ &= \frac{Ce^{-\alpha(x^2+y^2)}}{2\alpha} + f(y) \\ &= \frac{Ce^{-\alpha(x^2+y^2)}}{2\alpha} + f(y)\end{aligned}$$

$f(y)$ can be obtained by integrating $\frac{\partial V}{\partial y}$ or by differentiating the above expression. Using latter

$$\begin{aligned}\frac{\partial V}{\partial y} &= \frac{-2\alpha y Ce^{-\alpha(x^2+y^2)}}{2\alpha} + \frac{\partial f(y)}{\partial y} \\ &= -F_y \\ &= -Cye^{-\alpha(x^2+y^2)} \\ \implies f(y) &= \text{const} \\ \implies V &= \frac{Ce^{-\alpha(x^2+y^2)}}{2\alpha}\end{aligned}$$

upto a constant.

b)

$$\mathbf{F} = C(y\hat{\mathbf{x}} - x\hat{\mathbf{y}})e^{-\alpha(x^2+y^2)}$$

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \\ &= \left(-e^{-\alpha(x^2+y^2)} - x(-2\alpha x)e^{-\alpha(x^2+y^2)} - e^{-\alpha(x^2+y^2)} + y(-2\alpha y)e^{-\alpha(x^2+y^2)} \right) \hat{\mathbf{z}} \\ &= 2e^{-\alpha(x^2+y^2)}(\alpha x^2 + \alpha y^2 - 1) \\ &\neq 0\end{aligned}$$

\therefore potential energy function doesn't exist.

Note: The vector $y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$ is perpendicular to radius vector $x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. So the curl is non zero.

c)

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \\ &= C \left(y \left(-\beta \frac{2x}{2\sqrt{x^2+y^2}} \right) e^{-\beta\sqrt{x^2+y^2}} - x \left(-\beta \frac{2y}{2\sqrt{x^2+y^2}} \right) e^{-\beta\sqrt{x^2+y^2}} \right) \hat{\mathbf{z}} \\ &= 0\end{aligned}$$

\therefore potential energy function exists.

$$\begin{aligned}\mathbf{F} &= C(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})e^{-\beta\sqrt{x^2+y^2}} \\ &= -\nabla V\end{aligned}$$

$$\implies V = - \int C e^{-\beta\sqrt{x^2+y^2}} x dx$$

Put $x^2 + y^2 = r^2 \implies x dx = dr$. (Taking y as const)

$$\begin{aligned}V &= - \int C e^{-\beta r} r dr \\ &= -C \left[x \int e^{-\beta r} dr - \int \frac{dr}{dr} \left(\int e^{-\beta r} dr \right) dr \right] \\ &= -C \left[r \frac{e^{-\beta r}}{-\beta} - \int \frac{e^{-\beta r}}{-\beta} dr \right] \\ &= C \left[r \frac{e^{\beta r}}{\beta} + \frac{e^{-\beta r}}{\beta^2} \right] \\ &= C \frac{(\beta\sqrt{x^2+y^2} + 1)}{\beta^2} e^{-\beta\sqrt{x^2+y^2}} + f(y)\end{aligned}$$

Taking partial derivative of above expression wrt y ,

$$\begin{aligned}\frac{\partial V}{\partial y} &= C \left[\frac{(\beta 2y)}{2\sqrt{x^2+y^2}\beta^2} e^{-\beta\sqrt{x^2+y^2}} + \frac{(-2y\beta)(\beta\sqrt{x^2+y^2}+1)}{2\sqrt{x^2+y^2}\beta^2} e^{-\beta\sqrt{x^2+y^2}} \right] + \frac{\partial f(y)}{\partial y} \\ &= -Cye^{-\beta\sqrt{x^2+y^2}} + \frac{\partial f(y)}{\partial y} \\ &= -F_y \\ &= -Cye^{-\beta\sqrt{x^2+y^2}}\end{aligned}$$

$$\implies f(y) = \text{const}$$

$$\therefore V = C \frac{(\beta\sqrt{x^2+y^2}+1)}{\beta^2} e^{-\beta\sqrt{x^2+y^2}}$$

upto a constant

d)

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y & xy^2 & z \end{vmatrix} \\ &= (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y^2-x^3)\hat{\mathbf{z}} \\ &\neq 0\end{aligned}$$

\therefore potential energy function doesn't exist.

6.2

$$\mathbf{F} = -\mathbf{r}$$

At $t = 0 : x = 1, y = 0, v_x = 0, v_y = 2$

1)

At $t = 0, \hat{\mathbf{r}} = \hat{\mathbf{i}}, \hat{\phi} = \hat{\mathbf{j}}$

$$r(0) = \sqrt{x^2+y^2} = 1$$

$$\frac{dr(0)}{dt} = v_x = 0$$

$$\frac{d\phi(0)}{dt} = \frac{v_y}{r} = 2$$

2)

$F = -\nabla\left(\frac{x^2+y^2}{2}\right) = -\nabla(r^2/2) \implies F$ is conservative and $V = -r^2/2$. The force is radial, so the angular momentum is also conserved.

$$\begin{aligned} E &= \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} + V \\ &= \frac{\dot{r}^2}{2} + \frac{r^2\dot{\phi}^2}{2} + \frac{r^2}{2} \\ &= E(t=0) \\ &= 0 + 2 + .5 \\ &= 2.5 \\ L &= mr^2\dot{\phi} \\ &= r^2\dot{\phi} \\ &= L(t=0) \\ &= 2 \end{aligned}$$

Using angular momentum eqn in energy eqn gives,

$$\dot{r}^2 + \frac{4}{r^2} + r^2 = 5$$

4)

$$\begin{aligned} \dot{r}^2 &= 5 - \frac{4}{r^2} - r^2 \geq 0 \\ \implies r^4 - 5r^2 + 4 &\leq 0 \\ \implies (r^2 - 1)(r^2 - 4) &\leq 0 \\ \implies 1 \leq r \leq 2 \end{aligned}$$

5)

In cartesian coordinates:

$$\begin{aligned} \ddot{x} &= -\frac{x}{m} = -x \\ \ddot{y} &= -\frac{y}{m} = -y \\ \implies x &= A \sin t + B \cos t, y = C \sin t + D \cos t \end{aligned}$$

Initial conditions:

$$x(0) = 1 \implies B = 1, \dot{x}(0) = 0 \implies A = 0$$

$$\therefore x = \cos t$$

$$y(0) = 0 \implies D = 0, \dot{y}(0) = 2 \implies C = 2$$

$$\therefore y = 2 \sin t$$

$$x^2 + \frac{y^2}{4} = \sin^2 t + \cos^2 t = 1$$

This is equation of ellipse with length of major axis = 2 and that of minor axis = 1.

5.3

1)

$$\mathbf{F} = -\nabla(xy) = -y\mathbf{i} - x\mathbf{j}$$
$$m\ddot{x} = -y \implies \ddot{x} = -y \quad (1)$$

$$m\ddot{y} = -x \implies \ddot{y} = -x \quad (2)$$

2)

At $t = 0, x = 1, y = 1, \dot{x} = 0, \dot{y} = 0$. Adding and subtracting eqn (1) and (2) gives

$$\frac{d^2}{dt^2}(x + y) = -(x + y)$$
$$\implies x + y = A \sin t + B \cos t$$

$$\frac{d^2}{dt^2}(x - y) = (x - y)$$
$$\implies x - y = Ce^t + De^{-t}$$

Initial conditions:

$$x(0) + y(0) = 2 \implies B = 2, \dot{x}(0) + \dot{y}(0) = 0 \implies A = 0$$
$$\therefore x + y = 2 \cos t \quad (3)$$

$$x(0) - y(0) = 0 \implies C + D = 0, \dot{x}(0) - \dot{y}(0) = 0 \implies C - D = 0$$
$$\implies C + D = 0$$
$$\therefore x - y = 0 \implies x = y$$

Putting in eqn (3) gives

$$x = y = \cos t$$

3)

Since $\mathbf{F} = -\nabla U$, \mathbf{F} is conservative. Therefore total energy is conserved.

$$E = \frac{mv^2}{2} + U = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + xy$$
$$= \frac{(\sin^2 t + \sin^2 t)}{2} + \cos t \times \cos t$$
$$= \sin^2 t + \cos^2 t$$
$$= 1$$
$$= E(t = 0)$$
$$= \frac{0 + 0}{2} + 1 \times 1$$
$$= 1$$

4)

$$\begin{aligned}M &= \dot{x}\dot{y} + \frac{x^2 + y^2}{2} \\ \dot{M} &= \ddot{x}\dot{y} + \dot{x}\ddot{y} + x\ddot{x} + y\ddot{y} \\ &= -y\ddot{y} - x\ddot{x} + x\ddot{x} + y\ddot{y} \\ &= 0\end{aligned}$$

$$M(t=0) = 0 \times 0 + \frac{1^2 + 1^2}{2} = 1$$

Using solution of equation

$$\begin{aligned}M(t) &= -\sin t \times -\sin t + \cos t \times \cos t \\ &= 1 \\ &= M(t=0)\end{aligned}$$

e)

$$p = x + y, q = x - y$$

$$p + q = 2x, p - q = 2y$$

Differentiating twice,

$$\ddot{p} = \ddot{x} + \ddot{y}, \ddot{q} = \ddot{x} - \ddot{y}$$

Using eqn of motions form 1)

$$\ddot{p} = -y - x = -p$$

$$\ddot{q} = -y + x = q$$

$$\implies p = A \sin t + B \cos t \text{ and } q = C e^t + D e^{-t}$$

Initial conditions:

$$p(0) = x(0) + y(0) = 2 \implies B = 2, \dot{p}(0) = 0 \implies A = 0$$

$$q(0) = x(0) - y(0) = 0 \implies C + D = 0, \dot{q}(0) = 0$$

$$\implies C - D = 0 \implies C = D = 0$$

$$\begin{aligned}p &= 2 \sin t \\ &= \sin t + \sin t \\ &= x + y \\ q &= 0 \\ &= \sin t - \sin t \\ &= x - y\end{aligned}$$