Problem 2.1

$$y''' + y'' + y' + y = 0$$

Assume a solution of the form $y = e^{\alpha x}$

$$(\alpha^3 + \alpha^2 + \alpha + 1)e^{\alpha x} = 0$$

$$\Rightarrow (\alpha^3 + \alpha^2 + \alpha + 1) = 0$$

$$\alpha^2(\alpha + 1) + (\alpha + 1) = 0$$

$$(\alpha^2 + 1)(\alpha + 1) = 0$$

$\alpha = -1, \pm i$ is the solution. $\therefore y = A_1 e^{-x} + A_2 e^{ix} + A_3 e^{-ix}$ is the general sol of the equation. Use initial conditions:

$$y(0) = 0 \Rightarrow A_1 + A_2 + A_3 = 0$$

$$y'(0) = -1 \Rightarrow -A_1 + iA_2 - iA_3 = -1$$

$$y(0) = 2 \Rightarrow A_1 - A_2 - A_3 = 2$$

Solving the above simultaneous equation gives $A_1 = 1, A_2 = -1/2, A_3 = -1/2$

$$\therefore y = e^{-x} + (-1/2)e^{ix} + (-1/2)e^{-ix}$$

$$y = e^{-x} - \cos x$$

2.2 a) No this is not a linear equation since the term $1 - x^2$ is not linear.

b)Approx for RHS for $x << 1$=

$$2\beta \frac{dx}{dt}$$

c)

$$\frac{d^2x}{dt^2} + x = 2\beta \frac{dx}{dt}$$

Assume sol of the form $y = e^{ax}$

$$a^2 - 2\beta a + 1 = 0$$

$$\Rightarrow a = \beta \pm \sqrt{\beta^2 - 1}$$

If $\beta << 1$

$$a = \beta \pm i$$

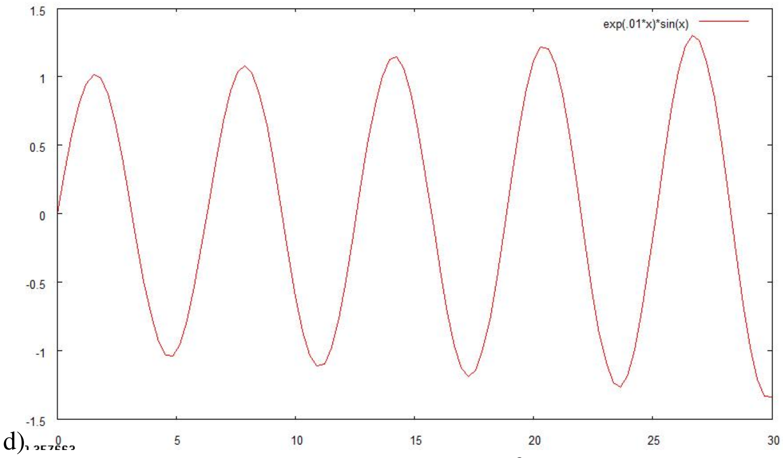
The general solution is

$$x = A_1 e^{(\beta + i)t} + A_2 e^{(\beta - i)t}$$

$$x(0) = 0 \Rightarrow A_1 + A_2 = 0 \Rightarrow x = A_1 e^{\beta t} \sin(t)$$

$$v(0) = 0.1 \Rightarrow A_1 \beta e^{\beta t} \sin(t) + A_1 e^{\beta t} \cos(t) = 0.1 \Rightarrow A_1 = 0.1$$

$$\therefore x = 0.1e^{\beta t}\sin(t)$$



d)

For small t, $x \approx .1(1+\beta t)t \approx 0.1t$ (expanding $e^{\beta t}$and $\sin(t)$ to first order terms)

$$x << 1 \Rightarrow 0.1e^{\beta t}\sin(t) << 1 \Rightarrow t << \frac{\ln(10)}{\beta}$$

2.3

$$x(\frac{d}{dx}\left[x\frac{dy}{dx}\right]) = -k^2 y$$

$$x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = -k^2 y$$

Assume sol of the form $y = x^n$. Putting it in eqn

$$n(n-1)x^n + nx^n = -k^2 x^n$$

$$n^2 = -k^2$$

$$n = \pm ik$$

Therefore,

$$y = A_1 x^{ik} + A_2 x^{-ik}$$

$$y = A_1 e^{ln(x^{ik})} + A_2 e^{ln(x^{-ik})}$$

$$y = A_1 e^{ikln(x)} + A_2 e^{-ikln(x)}$$

$$y = (A_1 + A_2)\cos(k\ln(x)) + i(A_1 - A_2)\sin(k\ln(x))$$

$$s = ln(x) \Rightarrow x = e^s$$

.

$$\frac{df}{ds} = \frac{dx}{ds}\frac{df}{dx} = e^s\frac{df}{dx} = x\frac{df}{dx}$$

where f is an arbitrary function. Therefore the equation becomes,

$$\frac{d^2y}{ds^2} = -k^2 y$$

$$y = A_3 \sin(ks) + A_4 \cos(ks)$$

$$y = A_3 \sin(k\ln(x)) + A_4 \cos(k\ln(x))$$

The solutions obtained are the same except for the arbitrary consts.

2.4 a)

$$m\frac{dv}{dt} = -g$$

$$\Rightarrow v = -gt + c_1$$

Putting $v(0) = v_0$ gives $c_1 = v_0$

$$\frac{dx}{dt} = -gt + v_0$$

$$x = \frac{-gt^2}{2} + v_0 t + c_2$$

Putting $x(0) = 0$ gives $c_2 = 0$

$$\therefore x = \frac{-gt^2}{2} + v_0$$

b)Energy method:$f(x) = -g$

$$\Rightarrow U = gx$$

$$\Rightarrow \frac{d}{dt}(\frac{\dot{x}^2}{2}) = \frac{d}{dt}(-gx)$$

$$\frac{\dot{x}^2}{2} + gx = C$$

Put $\dot{x}(0) = v_0$ gives $C = v_0^2/2$

$$\dot{x} = \left[v_0^2 - 2gx\right]^{\frac{1}{2}}$$

$$\int dt = \int \frac{dx}{\left[v_0^2 - 2gx\right]^{\frac{1}{2}}}$$

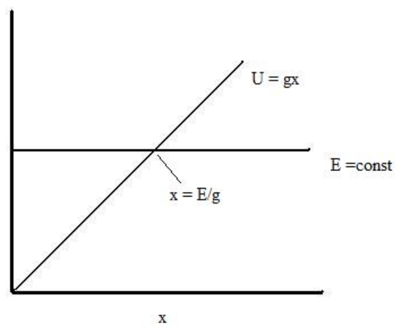
$$t = -\left(v_0^2 - 2gx\right)^{\frac{1}{2}} \frac{1}{g} + C$$

Using $x(0) = 0$ gives $C = \frac{v_0}{g}$

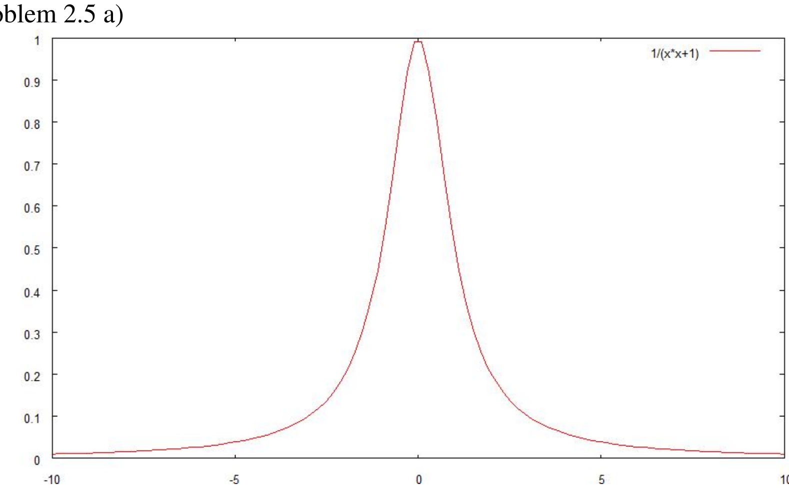
$$(gt - v_0)^2 = v_0^2 - 2gx$$

$$x = -\frac{1}{2g} gt(gt + 2v_0)$$

$$x = -\frac{gt^2}{2} + v_0 t$$



Problem 2.5 a)



The plot is for $U_0 = 1 \& b = 1$

b) The potential energy is max at $x = 0$. If particle is able to reach $x = 0$. Then from energy conservation

$$K.E(x = -\infty) + P.E(x = -\infty) = K.E(x = 0) + P.E(x = 0)$$

$$\frac{m}{2} \times \frac{U_0}{5m} + 0 = \frac{mv^2}{2} + \frac{U_0 b^2}{b^2}$$

$$mv^2/2 = -\frac{9U_0}{10}$$

but $v^2 \geq 0$. Therefore the particle can't reach positive values of x. At the farthest position on the right $v = 0$. Therefore from energy conservation

$$mv(x = -\infty)^2/2 = U(x)$$

$$\frac{U_0}{10} = \frac{U_0 b^2}{x^2 + b^2}$$
$$x^2 + b^2 = 10b^2$$
$$x = \pm 3b$$

Therefore $x = -3b$, as the particle can't reach positive x values.

c)

$$F = -\frac{dU}{dx}$$
$$F = \frac{U_0 b^2}{(x^2 + b^2)^2} \times 2x$$
$$F = \frac{2xU_0 b^2}{(x^2 + b^2)^2}$$

Put $x = -7$,

$$F = \frac{-14U_0 b^2}{(49 + b^2)^2}$$

The direction of the force is away from the origin.

2.6

$$F = -\frac{dU}{dx} = -\frac{C}{x^3}$$
$$U = \frac{-C}{2x^2} \text{ (Ignoring the constant)}$$

From energy conservation

$$mv(x)^2/2 - \frac{C}{2x^2} = \frac{-C}{2x_0^2}$$
$$v(x) = \left[\frac{C}{m}\left(\frac{1}{x^2} - \frac{1}{x_0^2}\right)\right]^{\frac{1}{2}}$$
$$\int dt = \int \frac{-dx}{\left[\frac{C}{m}\left(\frac{1}{x^2} - \frac{1}{x_0^2}\right)\right]^{\frac{1}{2}}}$$

$\because$ x decreases with time. The ambiguity is because of the squareroot.

$$t = \sqrt{\frac{m}{c}} x_0^2 x \sqrt{\frac{1}{x^2} - \frac{1}{x_0^2}} + C$$
$$t = \sqrt{\frac{m}{c}} x_0^2 \sqrt{1 - \frac{x^2}{x_0^2}} + C$$

Using $x = x_0$ at $t = 0$ gives $C = 0$

t for $x = 0$

$$t = \sqrt{\frac{m}{c}} x_0^2$$

Dimensional Analysis: Dim of $C = Force \times L^3 = ML^3T^{-2}$,$m = M$, $x_0 = L$

$$T = M^a L^b (ML^3T^{-2})^c$$

Solving for a, b, c gives

$$t = \sqrt{\frac{m}{C}} x_0^2$$