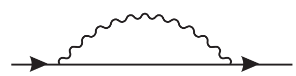
Take-home mid-term exam

1. Electrons in a metal interact with the crystal lattice. The corresponding electron-phonon interaction Hamiltonian is

$$\mathcal{H}_{\text{int}} = g \int \psi^\dagger_\alpha(\mathbf{r})\psi_\alpha(\mathbf{r})\phi(\mathbf{r})d^3\mathbf{r},$$

where g is a small constant, ψ and $\psi^\dagger$ are the electron field operators, and ϕ is the phonon field operator. Consider the following diagram for the electron self-energy.

$$\Sigma =$$



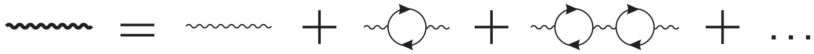
Here the solid line corresponds to the electron Green's function and the wavy line to the phonon propagator. Using, the Debye model for phonons...

   (a) Calculate the imaginary part of the self-energy $\Sigma(\varepsilon, p)$ in two limits: $|\varepsilon| \ll \omega_{\text{D}}$ and $|\varepsilon| \gg \omega_{\text{D}}$, where ω_D is the Debye frequency.

   (b) Calculate the real part of the self-energy and find the correction to the electron effective mass.

2. Consider longitudinal phonons in the Debye model. Calculate the heat capacity of a phonon gas in the high-temperature and low-temperature limits.

3. The screening of Coulomb interaction in an electron gas is described by the following diagram series,



where the dashed line is the Coulomb interaction and the bubble corresponds to the polarization operator

$$\Pi(\omega, \mathbf{q}) = 2i \int G\left(\varepsilon + \frac{\omega}{2}, \mathbf{p} + \frac{\mathbf{q}}{2}\right) G\left(\varepsilon - \frac{\omega}{2}, \mathbf{p} - \frac{\mathbf{q}}{2}\right) \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d\varepsilon}{2\pi},$$

where $G(\varepsilon, \mathbf{p})$ is the free electron Green's function and d is the dimensionality of the system.

The dynamically screened Coulomb interaction is described by the following formula

$$V(\omega, \mathbf{q}) = \frac{v(\mathbf{q})}{1 - v(\mathbf{q})\Pi(\omega, \mathbf{q})},$$

where $v(\mathbf{q})$ is the bare Coulomb potential.

Consider a *two-dimensional* ($d = 2$) electron gas and...

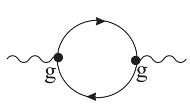
(a) Calculate the bare (unscreened) Coulomb interaction in momentum space, by Fourier-transforming the potential $v(r) = e^2/r$.

(b) Calculate the polarization operator at $\omega = 0$ and $\mathbf{q} = \mathbf{0}$, and find the analog of the Debye screening length in two dimensions (write the screened Coulomb potential at $q = 0$ and $\omega = 0$).

(c) Calculate the screened Coulomb potential in real space.

(d) Calculate the polarization operator at small but finite frequencies and momenta $\omega \ll E_{\mathrm{F}}$ and $q \ll p_{\mathrm{F}}$. Find the spectrum of collective modes (two-dimensional plasmons). This spectrum is the solution of the equation $v(\mathbf{q})\Pi\left[\omega(\mathbf{q}), \mathbf{q}\right] = 1$. What is the main difference between the two-dimensional and three-dimensional plasmons?

4. The leading order correction to the phonon propagator is given by the diagram (*c.f.*, problem 3 above),



$$\Longrightarrow \qquad D^{-1}(\omega,\mathbf{k}) = D_0^{-1}(\omega,\mathbf{k}) + g^2\,\Pi(\omega,\mathbf{k})$$

where the wavy line corresponds to the phonon propagator, the expression for the bubble is given above, and each vertex corresponds to the electron-phonon coupling constant g.

Find the leading order correction to the speed of sound in three dimensions. Assume that the bare speed of sound is much smaller than the Fermi velocity, $c \ll v_{\mathrm{F}}$.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii, Mahan, and Lectures

Due Thursday, March 15 (in class) Each student gives a 15-20 minute presentation on one of the problems.