# Take-home final exam

1. Thermodynamic potential of an ideal Fermi gas can be found from the following formula

$$\Omega = -T \int \ln\left(1 + e^{-\epsilon(\mathbf{p})/T}\right) \frac{d^d\mathbf{p}}{(2\pi)^d},$$

   where $d$ is the dimensionality of the system. Starting from this expression, prove that the specific heat is a linear function of temperature at $T \ll E_{\mathrm{F}}$. Calculate the proportionality constant $\gamma$ in three and two dimensions, $C = \gamma T$.

2. Consider the following Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mathbf{p},\sigma} \frac{\mathbf{p}^2}{2m} \hat{c}^\dagger_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} + \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) \hat{\rho}_{\mathbf{q}} \hat{\rho}_{-\mathbf{q}} + \sum_{\mathbf{q}} V_{\mathrm{ext}}(\mathbf{q}) \hat{\rho}_{\mathbf{q}},$$

   where $\hat{c}$ and $\hat{c}^\dagger$ are electron annihilation and creation operators, $\hat{\rho}_{\mathbf{q}} = \sum\limits_{\mathbf{p},\sigma} \hat{c}^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}\sigma} \hat{c}_{\mathbf{p}-\frac{\mathbf{q}}{2}\sigma}$ is the electron density operator, $V(q)$ is the two-particle interaction, and $V_{\mathrm{ext}}(q)$ is the external potential.

   The current operator is defined as

$$\hat{\mathbf{J}}_{\mathbf{q}} = \sum_{\mathbf{p},\sigma} \frac{\mathbf{p}}{m} \hat{c}^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}\sigma} \hat{c}_{\mathbf{p}-\frac{\mathbf{q}}{2}\sigma}$$

   Prove the following operator identity (the continuity equation):

$$\frac{\partial \hat{\rho}_{\mathbf{q}}}{\partial t} - i\mathbf{q} \cdot \hat{\mathbf{J}}_{\mathbf{q}} = \hat{0}$$

3. For a one-dimensional harmonic oscillator with mass $m$ and frequency $\omega$, calculate the time-ordered $G_{AB}$ and temperature (Matsubara) $\mathcal{G}_{AB}$ Green's functions for the following choices of the operators $\hat{A}$ and $\hat{B}$:

   (a) Both $\hat{A}$ and $\hat{B}$ are equal to the position operator $\hat{x}$.

   (b) The operator $\hat{A}$ is equal to the position operator $\hat{x}$ and $\hat{B}$ is equal to the momentum operator $\hat{p}$.

   (c) The operator $\hat{A}$ is equal to the annihilation operator $\hat{a}$ and $\hat{B}$ is equal to the creation operator $\hat{a}^\dagger$.

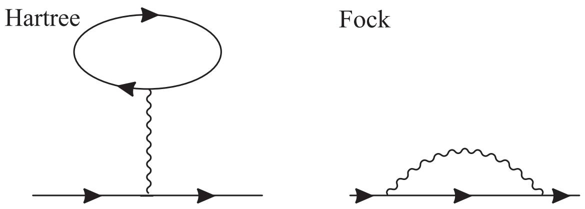
   The definitions of the Green's functions are reminded below

   - Time-ordered, $G_{AB}(t,t') = -i\left\langle T_t\left(\hat{A}(t)\ \hat{B}(t')\right)\right\rangle$

- Matsubara, $\mathcal{G}_{AB}(\tau,\tau') = -\left\langle T_\tau\left(\hat{A}(\tau)\ \hat{B}(\tau')\right)\right\rangle$

In above definitions $A(t)/A(\tau)$ are Heisenberg operators in real/imaginary time, $T_t/T_\tau$ are real/imaginary time-ordering operators, and $\langle\ldots\rangle$ indicates the quantum-mechanical averaging.

4. Consider a three-dimensional system of fermions, which interact with each other with a point-like potential $V(\mathbf{r}-\mathbf{r}') = u_0\delta(\mathbf{r}-\mathbf{r}')$. Calculate the fermionic self-energy in the Hartree-Fock approximation. I.e., calculate the following diagrams



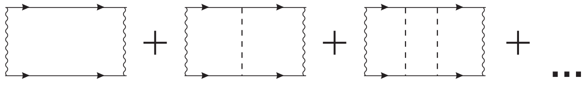
Within the Hartree-Fock approximation, derive the general formula for the correction to the chemical potential in terms of the spin $s$, the fermion density $n$, and the interaction strength $u_0$. Note that in the model of "spinless fermions," the correction vanishes. Can this fact be understood without calculations?

Hint: Note that the integral of the Green's function over energy is simply the (Fermi) distribution function.

5. Consider a three-dimensional system of fermions interacting with each other via a long range potential $v(r) = g^2/r^2$, where $r$ is the distance between the particles and $g$ is a small constant. Doing the standard RPA perturbation theory find the screened potential in real space and calculate the spectrum of collective modes.

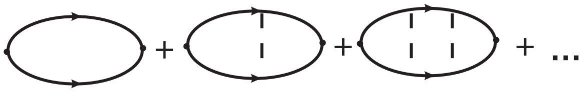
Hint: This problem is very similar to problem 3 of your mid-term (screening of Coulomb interaction in two dimensions).

6. Prove the Anderson theorem: Weak disorder does not affect the transition temperature, $T_\mathrm{c}$ in an $s$-wave superconductor. To determine $T_\mathrm{c}$, consider the Cooper ladder or Cooperon (see lectures) and average it over disorder realizations. Diagrammatically the Cooperon has the form



where the wavy line corresponds to the attractive BCS interaction [assume that it is a delta function in real space $V(\mathbf{r}-\mathbf{r}') = \lambda\delta(\mathbf{r}-\mathbf{r}')$] and the dashed line is the disorder potential (which is also a $\delta$-function in real space; the dashed line corresponds to $(2\pi\nu\tau)^{-1}\delta(\mathbf{r}-\mathbf{r}'))$.

To find the transition point, we need to calculate two geometric series: One due to the interaction ladder and the other due to the disorder ladder(see Fig. 3 below)



Thus the transition point is given by the poles of the following expression

$$\Gamma = -\frac{1}{\lambda^{-1}+\Pi_c}$$

where

$$\Pi_c = -T\sum_{\varepsilon_n}\frac{B_c(\varepsilon_n)}{1-B_c(\varepsilon_n)/(2\pi\nu\tau)}$$

and

$$B_c(\varepsilon_n) = \int\frac{d^dp}{(2\pi)^d}\mathcal{G}(\varepsilon_n,\mathbf{p})\mathcal{G}(-\varepsilon_n,-\mathbf{p})$$

with $\mathcal{G}(\varepsilon_n,\mathbf{p})$ being the Matsubara Green's function averaged over disorder potential (see AGD and lectures). Calculate the integrals, find the transition temperature, and prove that it is independent of the impurity concentration as long as disorder is weak (Anderson theorem).

7. Consider a scalar field theory in four dimensions defined by the following action

$$S[\phi,\phi^*] = S_0[\phi,\phi^*] + S_I[\phi,\phi^*],$$

where

$$S_0[\phi,\phi^*] = -\int\phi^*(\mathbf{k})(m^2+k^2)\phi(\mathbf{k})\frac{d^4k}{(2\pi)^4}$$

and

$$S_I[\phi,\phi^*] = -\frac{\lambda}{4}\int\phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\prod_{i=1}^{4}\frac{d^4k_i}{(2\pi)^4}$$

Derive the RG equation for the quartic coupling $\lambda$, by directly computing the averages in the cumulant expansion at one-loop, $(1/2)\left[\left\langle S_I^2\right\rangle - \left\langle S_I\right\rangle^2\right]$. Verify the combinatorial factors in Eq. (93) of Rev. Mod. Phys. **66**, 129 (1994); Also, see lectures.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii; Mahan; R. Shankar, Rev. Mod. Phys. **66**, 129 (1994); and Lectures

*Due Tuesday, May 15 (in class)* Each student gives a 15-20 minute presentation on one of the problems.