

Web page: <http://www.wam.umd.edu/~galitski/PHYS625/>

Do not forget to write your name and the homework number!

Please, staple all pages together.

Many-body Green's function; Basic of the Feynman diagram technique

1. Fermionic Green's function is defined (as zero temperature) as

$$G_{\alpha\beta}(x; x') = -i \langle T (\hat{\psi}_{H\alpha}(x) \hat{\psi}_{H\beta}^\dagger(x')) \rangle,$$

where ψ_H and ψ_H^\dagger are field operators in the Heisenberg representation, $\alpha, \beta = \uparrow, \downarrow$ are the spin indices, and $x = (t, \mathbf{r})$.

- (a) Prove that the density of fermions can be related to the Green's function as

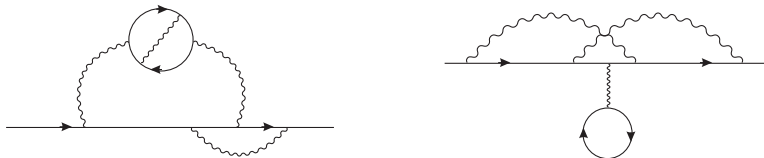
$$n(x) = -i \sum_{\alpha=\uparrow,\downarrow} \lim_{t' \rightarrow t+0} G_{\alpha\alpha}(t, \mathbf{r}; t', \mathbf{r}')$$

- (b) Using this formula for the density and the Green's function in momentum representation $G(\varepsilon, \mathbf{p})$, find the Fermi-momentum of free spinless fermions with the density n .
2. Electrons in a two-dimensional quantum well are often described by the Rashba model, defined by the following Hamiltonian

$$\hat{\mathcal{H}}_{\text{Rashba}} = \sum_{\mathbf{p}; \alpha, \beta} \left(\frac{\mathbf{p}^2}{2m} \delta_{\alpha\beta} + \lambda \mathbf{e}_z \cdot [\boldsymbol{\sigma}_{\alpha\beta} \times \mathbf{p}] \right) \hat{a}_{\mathbf{p}\alpha}^\dagger \hat{a}_{\mathbf{p}\beta},$$

where λ is a constant (spin-orbit coupling), \mathbf{e}_z is a unit vector in the z -direction, and $\boldsymbol{\sigma}_{\alpha\beta}$ are the Pauli matrices. Note that \mathbf{p} is a *two-dimensional* vector. Diagonalize the Hamiltonian and find the Green's function for the non-interacting Rashba electrons.

3. Draw all possible topologically non-equivalent diagrams (connected and disconnected) in second order perturbation theory with respect to a two-particle interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$.
4. Write down the analytical expressions (in momentum representation) corresponding to the following diagrams:



5. A localized magnetic impurity is introduced in a Fermi gas. The spin interacts with electrons via the following exchange interaction

$$\hat{\mathcal{H}}_{\text{int}} = J \int S^i \delta(\mathbf{r}) \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \sigma_{\alpha\beta}^i \hat{\psi}_{\beta}(\mathbf{r}) d^3\mathbf{r} \equiv JS \cdot \hat{\mathbf{S}}_{\text{el}}(\mathbf{r} = \mathbf{0}),$$

where J is a constant, which is assumed *small*, \mathbf{S} is the impurity spin, the index $i = x, y, z$ denotes spatial directions, α and β are the spin indices, and $\sigma_{\alpha\beta}^i$ are the Pauli matrices. The operator $\hat{\mathbf{S}}_{\text{el}}(\mathbf{r})$ denotes the *electron* spin density in the point \mathbf{r} . Find the average electron spin polarization, $\langle \hat{\mathbf{S}}_{\text{el}}(\mathbf{r}) \rangle$ at large distances from the localized magnetic impurity ($rp_{\text{F}} \gg 1$) in leading order of perturbation theory with respect to the exchange interaction.

Hint: Express the polarization density through the Green's function as follows $\langle \hat{\mathbf{S}}_{\text{el}}(\mathbf{r}) \rangle = \lim_{t' \rightarrow t+0} [-i \sigma_{\alpha\beta} G_{\beta\alpha}(\mathbf{r}, t; \mathbf{r}', t')]$. Calculate the first order correction to the Green's function $G_{\alpha\beta}^{(1)}(\varepsilon, \mathbf{r}; \mathbf{r}') = JS^i \sigma_{\alpha\beta}^i G_0(\varepsilon, \mathbf{r}) G_0(\varepsilon, -\mathbf{r}')$ in real space.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii

Due Monday, April 19 (in class)