Homework #2 — PHYS 625 — Spring 2017 Professor Victor Galitski
Deadline: Monday, March 13, 2017, in class Office: 2330 Physics

Relevant textbook: Abrikosov, A.A., Gor'kov, L.P., and Dzyaloshinskii, I.Ye., *Methods of Quantum Field Theory in Statistical Physicss*, Dover Publications Inc, New York, ISBN-10: 0486632288

Web page: `http://terpconnect.umd.edu/~galitski/PHYS625/`

Do not forget to write your name and the homework number!

Schrödinger equation in momentum space. Green function

1. Read Chapters 6 and 7 of the textbook by Abrikosov *et al.*

2. It was shown in class that a weak short-range potential (modelled as a δ-potential) always hosts a bound state in dimensions one and two, but has no bound state in three dimensions. Interestingly, the weakly-bound state "reappears" in a 3D many-fermion system and leads to formation of Cooper pairs (fermionic bound states) in the presence of an arbitrarily weak attraction between fermions. This fact is central to understanding superconductivity. It was realized in the 1950s that phonons are responsible for providing a "pairing glue" (the weak attractive potential) for electrons in metals. The simplest model of phonon-mediated superconductivity involves electrons near the Fermi surface interacting with each other via the attractive potential $V(\mathbf{p}_1, \mathbf{p}_2) = -\lambda$, if the momenta of both electrons lie within a thin energy shell near the Fermi surface $|E(\mathbf{p}_i) - E_\mathrm{F}| < \hbar\omega_\mathrm{D}$, with $\hbar\omega_\mathrm{D} \ll E_F$ being the Debye frequency. Otherwise, the interaction is zero. Use the Schrödinger equation in momentum space to find the binding energy of a Cooper pair that this interaction gives rise to.

Hint: You may follow the paper, "Bound Electron Pairs in a Degenerate Fermi Gas," Phys. Rev. **104**, 1189 (1956) by Leon Cooper for which he was awarded the 1972 Nobel prize in physics.

3. Fermionic Green's function is defined (as zero temperature) as

$$G_{\alpha\beta}(x; x') = -i \left\langle T\left(\hat{\psi}_{H\alpha}(x)\hat{\psi}^\dagger_{H\beta}(x')\right)\right\rangle,$$

where $\hat{\psi}_H$ and $\hat{\psi}^\dagger_H$ are the Heisenberg field operators, α and β are the spin indices, and $x = (t, \mathbf{r})$.

Prove that the current density can be related to the Green's function as

$$\mathbf{j}(\mathbf{r}) = -\frac{\hbar}{2m} \lim_{\mathbf{r}' \to \mathbf{r}} \lim_{t' \to t+0} \left(\boldsymbol{\nabla}_\mathbf{r} - \boldsymbol{\nabla}_{\mathbf{r}'}\right) G_{\alpha\alpha}(t, \mathbf{r}'; t', \mathbf{r}'). \tag{1}$$

4. A localized magnetic impurity is introduced in a Fermi gas. The spin interacts with electrons via the following exchange interaction

$$\hat{\mathcal{H}}_{\text{int}} = J \int S^i \delta(\mathbf{r}) \hat{\psi}^\dagger_\alpha(\mathbf{r}) \hat{\sigma}^i_{\alpha\beta} \psi_\beta(\mathbf{r}) d^3\mathbf{r} \equiv J S^i \hat{\sigma}^i_{\text{el}}(\mathbf{r} = \mathbf{0}), \tag{2}$$

where J is a constant, which is assumed *small*, $\mathbf{S}$ is the impurity spin, $i = x, y, z$, α and $\beta = \uparrow, \downarrow$ are spin indices, $\sigma^i_{\alpha\beta}$ are the Pauli matrices, and $\hat{\sigma}^i_{\text{el}}(\mathbf{r})$ is the position-dependent electron spin density operator. Find the spin polarization $\langle \hat{\sigma}^i(\mathbf{r}) \rangle$ at large distances from the localized magnetic moment (specify, what "large distances" mean in the given problem).

Hint: It will be proven in class that the electron Green's function can be represented in first-order perturbation theory as follows:

$$G^{(1)}_{\alpha\beta}(\varepsilon, \mathbf{r}; \mathbf{r}') = G_0(\varepsilon, \mathbf{r}) \delta_{\alpha\beta} + J S^i \sigma^i_{\alpha\beta} G_0(\varepsilon, \mathbf{r}) G_0(\varepsilon, -\mathbf{r}'). \tag{3}$$

Express the spin polarization density through the Green's function as

$$\left\langle \hat{\sigma}^i_{\text{el}}(\mathbf{r}) \right\rangle = \lim_{t' \to t+0} \left[-i \sigma^i_{\alpha\beta} G_{\beta\alpha}(\mathbf{r}, t; \mathbf{r}', t') \right]$$

and use the Green's function $G^{(1)}_{\alpha\beta}(\varepsilon, \mathbf{r}; \mathbf{r}')$ of Eq. (3) to calculate the spin density distribution.

Hint: The density oscillations that you found here are directly related to the so-called RKKY interactions, which is a leading coupling mechanism between nuclear magnetic moments in metals. See, *e.g.*, en.wikipedia.org/wiki/RKKY_interaction