Homework #2 — PHYS 625 — Spring 2016 | Professor Victor Galitski
Deadline: Wednesday, March 30, 2016, in class | Office: 2330 Physics

Relevant textbook (for problems 1 and 2): Al. Altland and B. Simons , Condensed Matter Field Theory, 2nd Edition, Cambridge University Press (2010)

Web page: `http://terpconnect.umd.edu/~galitski/PHYS625/`

Do not forget to write your name and the homework number!

Topics: Feynman path integral. Statistical transmutation. Toy version of diagrammatic technique

1. [1] Derive the Feynman path integral expansion for the partition function of a particle with the Hamiltonian, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. The partition function is defined in the usual way

$$Z(T) = \mathrm{Tr}\, e^{-\beta \hat{H}}, \text{ where } \beta = 1/(k_{\mathrm{B}}T)$$

2. [1] The transition amplitude of a particle with the Hamiltonian, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$$G(x_f, x_i; t) = \langle x_f | e^{-\frac{i}{\hbar}\hat{H}t} | x_i \rangle$$

can be written in terms of a Feynman path integral as follows

$$G(x_f, x_i; t) = \int\limits_{x(0)=x_i}^{x(t)=x_f} \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]},$$

where $S[x(t)]$ is a classical action of the particle following the trajectory, $x(t)$. Use the Feynman representation to derive an explicit expression for $G_0(x_f, x_i; t)$ of the free particle (i.e., $V(x) \equiv 0$) in one dimension.

What relation, if any, the function $G_0(x_f, x_i; t)$ has with the Green function of the single-particle Schrödinger equation, as discussed in lectures?

3. The Jordan-Wigner transformation is used to convert the spin operators into fermion operators (to "fermionize" spins). It has the following explicit form (see lecture notes) for spins/fermions on a one-dimensional lattice (n and m below label discrete lattice sites):

$$\hat{\sigma}_n^+ = \hat{f}_n^\dagger \prod_{m<n} \hat{\sigma}_m^z$$

[1] You can use any book you like to look up the derivation, but should be able to understand all steps you write down and be ready to answer questions about those. See also your lecture notes.

and

$$\hat{\sigma}_n^- = \hat{f}_n \prod_{m<n} \hat{\sigma}_m^z,$$

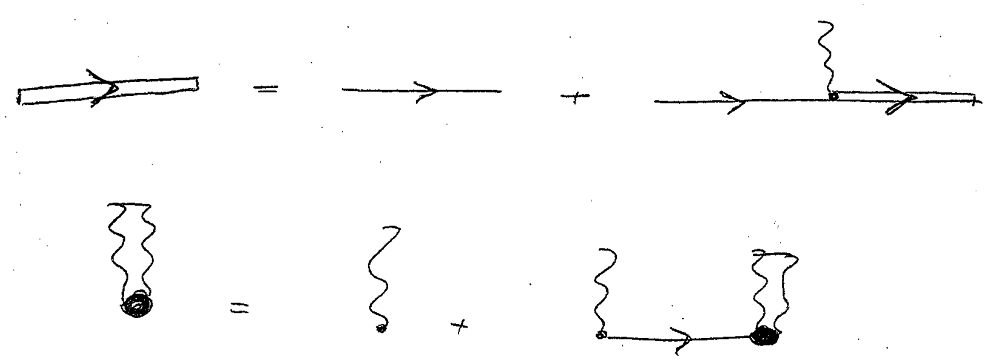
where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ is the vector of Pauli matrices, $\hat{\sigma}^\pm$ are the raising/lowering operators for spin, and $\hat{f}^\dagger$ and $\hat{f}$ are fermion creation/annihilation operators.

We have seen (see lecture notes) that Jordan-Wigner transform is particularly useful for solving a quantum spin chain with nearest neighbour spin-spin interactions. However, it turns out to be not as useful if longer-range interactions are present. Consider a quantum spin-1/2 chain with nearest-neighbour (labelled as $\langle nm \rangle$ below) and next-to-nearest-neighbour (labelled as $\langle\langle nm \rangle\rangle$ below) interactions as follows

$$\hat{H} = J_1 \sum_{\langle nm \rangle} \hat{\sigma}_n^+ \hat{\sigma}_m^- + J_2 \sum_{\langle\langle nm \rangle\rangle} \hat{\sigma}_n^+ \hat{\sigma}_m^-$$

Express this Hamiltonian in terms of the Jordan-Wigner fermion operators and explain why the transform here is not as useful as in the nearest-neighbour case. However, if $J_1 = 0$ it becomes useful again, why?

4. [2)] In the lectures, we have derived the following equations (pictorially represented below)



for the Green function and the scattering amplitude of the single-particle Schrödinger equation, correspondingly. We also used these exact equations to derive the energy of a bound state in a shallow potential well in one and two dimensions. Consider a three-dimensional shallow potential well (that can be modelled as a δ-function) and prove that it contains no bound state in contrast to 1D and 2D.

[2)] This problem requires almost no calculations, but just arguments based on the derivation in the lecture.