Time-independent degenerate perturbation theory

1. Deformed 2D harmonic oscillator [8 points] Consider a harmonic oscillator in two dimensions with the (unperturbed) Hamiltonian as follows (see also your lecture notes)

\[ \hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\hat{p}_y^2}{2m} + \frac{m\omega^2 y^2}{2}. \]

A weak perturbation is added to the isotropic harmonic potential above

\[ \hat{V} = \alpha \hat{x} \hat{y}. \]

- Find the degeneracy (i.e., the number of linearly-independent wave-functions with the same eigen-energies) of the second, third, and forth excited states of the unperturbed system. [1 point]

- Using methods of degenerate perturbation theory, write explicitly a matrix, whose eigenvalues determine corrections to the energies of the second excited states (those, whose unperturbed energies are equal to \( \hbar \omega \)). [3 points]

- Calculate explicitly the corrections to the energy of the second excited state. Is the degeneracy lifted? [3 points]

- Estimate the value of \( \alpha \), where perturbation theory breaks down. [1 point]

2. Perturbation in an infinite quantum well [6 points]

A particle is confined in a three-dimensional infinite quantum well, which extends from 0 to \( a \) in all three directions, that is \( 0 \leq x, y, z \leq a \). The system is perturbed by the delta-function potential at point \( \mathbf{r} = (a/4, a/2, 3a/4) \) as follows

\[ V = V_0 a^3 \delta \left( x - \frac{a}{4} \right) \delta \left( y - \frac{a}{2} \right) \delta \left( z - \frac{3a}{4} \right). \]

Find first order corrections to the energy of the ground state and the first excited (three-fold degenerate) state.

4. **Zeeman effect [6 points]**

A hydrogen atom in an external magnetic field can be described by the Hamiltonian

\[ H = \frac{p^2}{2m} - \frac{e^2}{r} - \mu_B B (\hat{L}_z + 2\hat{S}_z), \]

where \( \mu_B \) is the Bohr magneton, \( B \) is the strength of magnetic field, and \( \hat{L} \) and \( \hat{S} \) are orbital and spin angular momentum operators. Find the first order correction to the energy of the excited states with principal quantum number \( n = 2 \) (there are four degenerate levels).

5. **Stark effect [8 points]**

A hydrogen atom in an external electric field can be described by the Hamiltonian

\[ H = \frac{p^2}{2m} - \frac{e^2}{r} - \mathcal{E} z, \]

where \( \mathcal{E} \) is the strength of electric field. Find the first order correction to the energy of the excited states with principal quantum number \( n = 2 \) (there are four degenerate levels).

Note: hydrogen atom wave-functions can be found in tables 4.2 and 4.6 of the textbook.

6. **Perturbation theory [8 points]** Make up and solve your own problem, whose solution relies on time-independent perturbation theory (non-degenerate or degenerate). That is, formulate a problem with some unperturbed Hamiltonian with a known spectrum (\( \hat{H}_0 \) of your choice) and some perturbation (\( \hat{V} \) of your choice) and solve this your problem using perturbation theory (here, by “to solve” I mean to find a correction to the energy of a state or states). You are welcome to collaborate on this question, but if you do so, please indicate the names of all students, who contributed to your answer and solution.