

Textbooks: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9
R. Shankar *Basic Training in Mathematics: A Fitness Program for Science Students*, Plenum Publishing Corporation, 1995, ISBN 9780306450365

Web page: <http://www.wam.umd.edu/~galitski/PHYS374/>

Do not forget to write your name and the homework number!

Fourier Analysis

1. Fourier transform of a periodically-continued parabolic function [6 points]

Calculate the Fourier coefficients a_n and b_n , given by Eqs. (15.7) and (15.8) in the textbook, for the following function:

$$f(x) = x^2, \quad -L \leq x \leq L \quad (1)$$

and continued periodically $f(x + 2L) = f(x)$.

Sketch the function $f(x)$ over several periods. Use the symmetry to explain why one of the sets of the Fourier coefficients a_n or b_n vanishes identically.

Optional part: Using Mathematica or a similar program, make plots of the sum (15.2) with the coefficients a_n and b_n that you calculated. Keep a finite number of terms in the sum and observe how the sum converges to the sawtooth wave as you increase the number of terms in the sum.

2. Fourier transform of the rectified wave [6 points]

Calculate the Fourier coefficients a_n and b_n , given by Eqs. (15.7) and (15.8) in the textbook, for the following function:

$$f(x) = \cos(\pi x/2L), \quad -L \leq x \leq L \quad (2)$$

and continued periodically $f(x + 2L) = f(x)$.

Sketch the function $f(x)$ over several periods. Use the symmetry to explain why one of the sets of the Fourier coefficients a_n or b_n vanishes identically.

Optional part: Using Mathematica or a similar program, make plots of the sum (15.2) with the coefficients a_n and b_n that you calculated. Keep a finite number of terms in the sum and observe how the sum converges to the rectified wave as you increase the number of terms in the sum.

3. Fourier transform of exponential functions

- (a) [6 points] Calculate Fourier transforms of the following functions:

$$f(x) = \Theta(x) e^{-ax}, \quad (3)$$

$$f(x) = \Theta(-x) e^{ax}, \quad (4)$$

$$f(x) = e^{-a|x|}, \quad (5)$$

where a is a positive real number, and $\Theta(x)$ is the step function:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \quad (6)$$

- (b) [3 points] Using your Fourier transform of the function (5), prove the following relation:

$$\lim_{a \rightarrow 0} \int dx e^{-ixk} e^{-a|x|} = 2\pi \delta(k), \quad (7)$$

i.e. the Fourier transform of a constant function (when $a = 0$) is the delta-function.

4. Response function of an RL circuit

Consider a circuit consisting of an inductance L and a resistance R connected in series. The input voltage $V_{in}(t)$ is applied to the series of L and R , whereas the output voltage $V_{out}(t)$ is taken from the resistor R . The circuit equations are

$$V_{in}(t) = L \frac{dI(t)}{dt} + RI(t) \quad (8)$$

$$V_{out}(t) = RI(t) \quad (9)$$

- (a) [3 points] Equations (8) and (9) are written in the time domain. Fourier-transform these equations to the frequency domain, i.e. obtain equations relating the Fourier transforms

$$\tilde{V}_{in}(\omega) = \int dt e^{i\omega t} V_{in}(t), \quad \tilde{V}_{out}(\omega) = \int dt e^{i\omega t} V_{out}(t), \quad \tilde{I}(\omega) = \int dt e^{i\omega t} I(t). \quad (10)$$

In particular, show that

$$\frac{dI(t)}{dt} \rightarrow -i\omega \tilde{I}(\omega). \quad (11)$$

- (b) [3 points] Eliminate $\tilde{I}(\omega)$ from the equations obtained in Part 4a and obtain a relation between the input and output voltages in the frequency domain:

$$\tilde{V}_{out}(\omega) = \tilde{G}(\omega) \tilde{V}_{in}(\omega), \quad (12)$$

where $\tilde{G}(\omega)$ is a response function. Derive an explicit expression for $\tilde{G}(\omega)$.

- (c) [3 points] Fourier-transform Eq. (12) to the time domain and obtain a relation between the input and output voltages in the form

$$V_{out}(t) = \int_{-\infty}^t dt' G(t-t') V_{in}(t'). \quad (13)$$

Derive an explicit expression for $G(t)$ using the results of Problem 3a. Make sure that dimensionality is correct in your final result.

5. Transmission line and filtering

- (a) [2 points] Suppose an input signal $V_{in}(t)$ is sent through an ideal transmission line, so the output signal $V_{out}(t)$ emerges with a delay time t_0 :

$$V_{out}(t) = V_{in}(t - t_0). \quad (14)$$

Show that, in this case, the response function has the following forms in the time and frequency domains:

$$G(t) = \delta(t - t_0), \quad \tilde{G}(\omega) = \int dt e^{i\omega t} G(t) = e^{i\omega t_0}. \quad (15)$$

- (b) [4 points] Suppose the input signal is a Gaussian function

$$V_{in}(t) = V_0 \exp(-t^2/2\tau^2). \quad (16)$$

This is a pulse of the width τ , which may represent, for example, a digital bit. Calculate the Fourier transform $\tilde{V}_{in}(\omega)$ of Eq. (16) to the frequency domain.

- (c) [2 points] Suppose the transmission line with a limited bandwidth in frequency (like a phone line). Let us model the limited bandwidth by a Gaussian function of ω with the width ω_0 and take the response function to be

$$\tilde{G}(\omega) = e^{i\omega t_0} \exp(-\omega^2/2\omega_0^2). \quad (17)$$

Using the result of Part 5b and Eq. (17), calculate the output signal $\tilde{V}_{out}(\omega)$ in the frequency domain:

$$\tilde{V}_{out}(\omega) = \tilde{G}(\omega) \tilde{V}_{in}(\omega). \quad (18)$$

- (d) [4 points] Fourier transform Eq. (18) to the time domain and obtain $V_{out}(t)$:

$$V_{out}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{V}_{out}(\omega). \quad (19)$$

What is the shape of the function $V_{out}(t)$? How does the width of $V_{out}(t)$ compare with the width of $V_{in}(t)$? Sketch both $V_{out}(t)$ and $V_{in}(t)$ on the same plot (do not forget the time delay t_0).

- (e) [2 points] Suppose we send a series of input pulses (16) with the time interval Δt between the pulses. The input pulses can be well resolved in $\Delta t \geq \tau$. Suppose we send the pulses through the transmission line with the limited bandwidth $\omega_0 \ll 1/\tau$. What is the lower limit on the time interval Δt between the pulses such that we can well resolve the pulses in the output? Observe that the bandwidth ω_0 of the transmission line limits how fast we can send digital pulses through.