Homework #9 — PHYS 374 — Fall 2007 Professor Victor Galitski
Deadline: **Tuesday, December 11, 2007, in class** Office: 2214 Physics

Textbook: Roel Snieder, *A Guided Tour of Mathematical Methods for the Physical Sciences*, Cambridge University Press, 2nd edition, 2004, ISBN 0-521-83492-9

Web page: http://www.wam.umd.edu/~galitski/PHYS374/

Do not forget to write your name and the homework number!

Residues; Complex integration

1. Find the residues of the following functions

- **(a)** [**2 points**] $(z^2+1)^{-2}$ at $z = \pm i$;
- **(b)** [**2 points**] $(z^2+z^3)^{-1}$ at $z = 0$ and $z = -1$;
- **(c)** [**2 points**] $\cos z/(z-\pi/2)^2$ at $z = \pi/2$;
- **(d)** [**2 points**] $\ln(1+3z)/z^3$ at $z = 0$;
- **(e)** [**2 points**] $e^{izt}/(iz+a^2)$ at $z = ia^2$ (where a and t are some positive constants).

2. Calculate the following integrals using the residue theorem.

- **(a)** [**3 points**]

$$I_1 = \int_0^\infty \frac{dx}{(1+x^2)^2};$$

- **(b)** [**3 points**]

$$I_2 = \oint_{S^1} \frac{dz}{z-a},$$

where $a = 1/2$ and S^1 is a circle of radius $R = 1$ (i.e., $|z| = 1$). Consider both orientations of the contour.

- **(c)** [**3 points**] The same integral as above but with $a = 2$.

3. The Green's function of the one-dimensional diffusion equation is defined by

$$\left(\mathcal{D}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) G(x-x', t-t') = \delta(x-x')\delta(t-t'), \tag{1}$$

where $\mathcal{D}$ is the diffusion coefficient. Find the solution of this equation [i.e., the explicit form of the Green's function $G(x,t)$] using the following steps (in your solution you can just set $x' = 0$ and $t' = 0$):

(a) [**5 points**] Introduce the Fourier transform of the Green's function as follows

$$\tilde{G}(q,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdt}{(2\pi)^2} G(x,t) e^{-iqx-i\omega t} \tag{2}$$

and find the equation for $\tilde{G}(q,\omega)$ by Fourier transforming Eq. (1).

(b) [**1 point**] Solve the corresponding *algebraic* equation and obtain the function $\tilde{G}(q,\omega)$ explicitly. The Green's function in real space-time can be found by calculating the inverse Fourier transform of $\tilde{G}(q,\omega)$.

(c) [**5 points**] First, calculate the inverse Fourier transform with respect to the frequency, ω, i.e.

$$\tilde{G}(q,t) = \int_{-\infty}^{\infty} d\omega G(q,\omega) e^{i\omega t}. \tag{3}$$

To evaluate this integral, you must use the contour integration method.

(d) [**5 points**] Finally, calculate the inverse Fourier transform with respect to the "wave-vector," q, and obtain the result.

Note, that the obtained Green's function describes random walk of a particle in one dimension. Namely, $G(x,t)$ is the probability of finding the particle in point x at the moment of time t, provided that initially the particle was in the origin, i.e., $x(t=0) = 0$.