Final Exam — PHYS 374 — Fall 2007 Professor Victor Galitski
Saturday, December 15, 2007 Office: 2214 Physics

You can use notecards and your lecture notes. You must not use books or any electronic devices (computers, cell phones, etc.).

Do not forget to write your name and staple the pages together!

Final exam

1. **Dimensional analysis; [10 points]** The Navier-Stokes equation, which describes the motion of a viscous fluid, has the form

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\cdot\mathbf{v}\right] = -\boldsymbol{\nabla} p + \mu\Delta\mathbf{v} + \mathbf{f}, \tag{1}$$

where ρ is the density, $\mathbf{v}$ is the velocity field, p is the pressure, μ is the viscosity coefficient, and $\mathbf{f}$ represents "other" forces *per unit volume*, such as gravity or centrifugal force. Find the physical dimension of the coefficient μ [**4 points**] and prove explicitly that all five terms in Eq. (1) have the same physical dimension [**6 points**].

2. **Perturbation theory/Taylor expansion [10 points]** Consider the following algebraic equation

$$x^2 - 3x - 4 = (1 - e^{\varepsilon})\cos x, \tag{2}$$

where $\varepsilon \ll 1$ is a small constant. Eq. (2) generally has two roots, which are functions of the parameter ε. Calculate the *positive* root $x(\varepsilon)$ up to the first order in ε. I.e., find the coefficients $x_{0,1}$ in the Taylor expansion of the function $x(\varepsilon) > 0$:

$$x(\varepsilon) = x_0 + x_1\varepsilon + \ldots. \tag{3}$$

3. **Gradient [10 points]** Consider a temperature field in two dimensions

$$T(x, y) = \sinh(x)\cos(x + y), \tag{4}$$

where $\sinh(x) = (e^x - e^{-x})/2$ is the hyperbolic sine function.

(a) [**1 point**] Is $T(x, y)$ a vector or scalar field? Is $(\operatorname{grad} T)$ a vector or scalar field?

(b) [**5 points**] Calculate $(\operatorname{grad} T)$ for arbitrary x and y.

(c) [**2 points**] Calculate $(\operatorname{grad} T)$ at the point $(\pi/8, \pi/8)$.

(d) [**2 points**] Calculate a unit vector in the direction tangent to the line of constant temperature at the point $(\pi/8, \pi/8)$.

4. **Dirac delta-function; [10 points]** Calculate the following integrals

(a) [**5 points**] $I_1 = \int\limits_0^\infty \cos x\, \delta\left(x^2 + \frac{\pi}{2}x - \frac{\pi^2}{2}\right) dx;$

(b) [**5 points**] $I_2 = \int\limits_{-\infty}^{\infty} \cos x \, \delta\left(x^2 + \frac{\pi}{2}x - \frac{\pi^2}{2}\right) dx.$

5. **Fourier transform; [10 points]** The temperature $T(x,t)$ in an infinitely long, thin rod satisfies the heat equation

$$\frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2}, \tag{5}$$

where λ is the heat conductivity. Suppose the temperature is given at time t $= 0$ by $T(x,0) = \theta(x) e^{-\alpha x}$, where $\theta(x)$ is the unit step function (0 for $x < 0$ and 1 for $x > 0$) and $\lambda > 0$.

   (a) [**5 points**] Find the Fourier transform $T(k,0)$ of $T(x,0)$.

   (b) [**5 points**] Find $T(x,t)$ at an arbitrary time t in terms of an integral over k; *Hint: Use the Fourier transform of Eq. (5) with respect to the position, x, but NOT the time, t.*

6. **Fourier transform; [10 points]** Develop the Fourier series representation for the variable-width square wave (*Note:* These types of functions are important in electronic music)

$$f(t) = \begin{cases} 1, & \text{if } t^2 < t_0^2; \\ 0, & \text{if } t^2 \geq t_0^2, \end{cases} \tag{6}$$

in the interval $[-\pi, \pi]$. In Eq. (6), t_0 is a constant such that $0 < t_0 < \pi$.

7. **Residue theorem; [15 points]** Evaluate the following integral

$$I(a) = \int\limits_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{a + \cos\phi}, \tag{7}$$

where a is an arbitrary constant. You should use the following steps:

   (a) [**1 point**] Express the cosine function in terms of the variable $z = e^{i\phi}$, where ϕ is the angle.

   (b) [**2 point**] Express Eq. (7) as an integral over the complex variable z by noticing that $dz = iz d\phi$ (prove it).

   (c) [**2 points**] Identify the contour of integration in the complex z-plane, which corresponds to the limits of integration in (7).

   (d) [**10 points**] Evaluate the complex integral using the residue theorem. (If you are unable to do it for arbitrary a [**10 points**], consider the limits of small a [**5 points**] and large a [**5 points**], i.e., $|a| \ll 1$ and $|a| \gg 1$).