Introduction to Game Theory IIIii

Payoffs: Probability and Expected Utility

Lecture Summary

• 1. Introduction
• 2. Probability Theory
• 3. Expected Values and Expected Utility.
1. Introduction

- We continue further discussion of characterizing payoffs in games.
- In many situations, players are confronted with uncertainty.
- We need to be able to describe their preferences over uncertain outcomes
  - The standard economic tool is Expected Utility.
- We also need tools to describe how they learn over the play of a game
  - The tool we use for this is Probability Theory.

2. Probability and Uncertainty
Probabilities

- We use the notion of probabilities to formalize the notion that many aspects of life are uncertain.
- To define probabilities, we need first to describe everything that might possibly occur (the “Sample Space”)
- The Sample Space is a set of elements called events or outcomes. For example, the sample space for flipping a coin is \{H,T\}
- The sample space for a simplified weather prediction might be \{sunny, cloudy, rainy, snowy\}.
- It is important to note that a sample space must include all possible outcomes. Therefore in the weather example, I have either ruled out sleet or included it in rainy.

Probabilities: Addition Rule

- This leads directly into a computation rule that partly helps us figure out the likelihood that sets of events occur. (That is, A OR B take place.)
- The probability that two or more non-overlapping (disjoint) sets of events occurs can be computed by adding up the individual probabilities.
- Example: Two (fair) coin tosses \(SS=\{HH,HT,TH,TT\}\). What is the probability of seeing at least one H?
Probabilities: Addition Rule

• There are three distinct events in which an H occurs, \(\{HH,HT,TH,TT\}\). The probability that an H occurs is \(.25+.25+.25=.75\). (Prob HH OR HT OR TH).
• The probability that a repeated H OR a repeated T occurs is \(.25+.25=.5\).
• Note: This rule ONLY works if the set of events are disjoint. (The event HH is distinct from any of the events, HT and TH.)
• The set of events, at least one H, and the set of events, first and last flips repeat are \(\{HH,HT,TH,TT\}\) and \(\{HH,TT\}\).
• What is the probability that either of these sets occur?

Modified Addition Rule

• Modified addition rule: If the two or more events are overlapping, we need to subtract the probability of the overlapping event to avoid double-counting.
• Example: The probability of drawing an ace is \(1/13=4/52\). The probability of drawing a spade is \(1/4 =13/52\). What is the probability of drawing an ace OR a spade?
Modified Addition Rule

• **Answer:** It is possible that when we draw a spade we draw the ace of spades. So adding $4/52+13/52=17/52$ would cause us to double count.
• The probability of drawing the ace of spades is $1/52$. Therefore, the correct probability is
• **Answer to past slide**
  • $\text{Prob[At least one H AND Repeat]}=1/4$. Therefore, $\text{Prob[At least one H OR Repeat]}=3/4+1/2-1/4=1$.
  • Why?

The Multiplication Rule

• A set of events, $A$ is independent of another set of events, $B$ if the probability $A$ occurs is the same whether or not $B$ is known to have occurred.
• The probability of multiple independent events is the product of the individual probabilities.
• **Example:** The probability that a head and then a tail occurs in two independent tosses of a fair coin is $.5*.5=.25$. The probability of drawing two hearts in a row (after replacing the first card) is $1/52*1/52$. 
Modified Multiplication Rule:

- Many sets of events are NOT independent, however.
- Suppose that the probability of rain given it is cloudy is $1/3$. (The conditional probability of rain given clouds.)
- We can write this as
  - $1/3 = \text{Prob}[\text{Rain} | \text{Cloud}].$
- Suppose that the unconditional probability of clouds is $1/2$.
  - That is, $1/2 = \text{Prob}[\text{Cloud}].$
- Suppose that the unconditional probability of rain is $1/4$.
  - That is, $1/4 = \text{Prob}[\text{Rain}].$
- Since $1/4$ is not equal to $1/3$, the two events, Rain and Cloudy are not independent.

Modified Multiplication Rule:

What is the probability that it both rains AND is cloudy?

$\text{Prob}[\text{Clouds AND Rain}] = \text{Prob}[\text{Clouds}] \times \text{Prob}[\text{Rain} | \text{Cloudy}]$

$= 1/2 \times 1/3 = 1/6.$

- **General:** If $X = Y$ AND $Z$,
  - $\text{Prob}[X] = \text{Prob}[Y] \times \text{Prob}[Z | Y].$
Modified Multiplication Rule: Bayes’ Rule

• Note that we can use this rule to derive another important rule -- Bayes’ Rule. This rule tells us the reverse conditional probability.
  – Prob[Cloudy | Rain]

• **Answer:** We need to know the unconditional probability of rain, say it is 1/4. We apply our earlier result two different ways to tell us:
  – Prob[Clouds AND Rain]=Prob[Clouds]*Prob[Rain | Clouds] * and
  – Prob[Clouds AND Rain]=Prob[Rain]*Prob[Clouds | Rain]

Modified Multiplication Rule: Bayes’ Rule

• We can combine these to get
  – Prob[Clouds]*Prob[Rain | Clouds]=Prob[Rain]*Prob[Clouds | Rain]

• Substituting in gives
  – 1/2*1/3 = 1/4 * Prob[Clouds | Rain] or
  – 2/3 = Prob[Clouds | Rain].

• That is, 2/3 of the time that it rains, it is cloudy.

• more generally

• This is the general form of Bayes’ Rule.
An Example of Using Bayes’ Rule

• Suppose you are a manager who hires a worker.
• The worker is drawn from a labor pool 25% of which are lazy and never work hard, 75% always work hard.
• You do not see how hard the worker works but, if he works hard, you get high profits with 80%.
• If he does not work hard, you get high profits with 20%.
• Suppose you experience high profits. What is the probability the worker is lazy?

• \[ \text{Answer: We are looking for } \Pr[\text{Lazy} | \text{High}] \].
• Bayes rule says
  - \[ \Pr[\text{H}] \cdot \Pr[\text{L} | \text{H}] = \Pr[\text{L}] \cdot \Pr[\text{H} | \text{L}] \].
• Using the data from the problem,
  - \[ .65 \cdot \Pr[\text{L} | \text{H}] = .25 \cdot .2 = .05 \].
  (Where does the 65% come from?)
• Therefore, the likelihood the worker is lazy has fallen to 7.7%. Is this reasonable?
• Should you fire the worker?
• Redo the problem with \[ \Pr[\text{H} | \text{NL}] = .6 \].
3. Expected Value and Expected Utility

- Suppose that a set of random events can be expressed as real numbers: Example -- SS={1,3,5}, and the associated probabilities are 1/4,1/3,5/12.
- The expected value (or mean or average) of these random events is the sum of the values each multiplied by its probability -- $EV=1/4*1+1/3*3+5/12*5 = 40/12$.
- **Example:** Suppose that a lottery offers a one in 5 million chance at $10 Million. The expected value of the lottery is $(1/5M)*$10M = $2.

**Expected Value**

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Expected Utility

• If people do not value money one for one, then the expected utility or value they place on a lottery will be different than the expected value of the lottery itself.
• **Example:** Suppose the utility from $10M is approximately 5M utility points and the utility from $1.50 is 1.25 utility points. Should I pay $1.50 for the lottery ticket?
• **Answer:** Expected utility from the ticket = \(1/5M \times 5M + 0 = 1 \text{ utility point}\)
• Expected utility from $1.50 (which I have for sure if I do not buy the ticket) = 1.25 utility points. Don’t buy the ticket.
• A consumer like this exhibits **risk aversion:** the expected utility from the gamble (1) is less than the expected value of the gamble (2).
• In Game Theory, we interpret payoffs in games in expected utility terms.

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**Expected Utility**

**Ranking gambles (or lotteries)**

<table>
<thead>
<tr>
<th>Gamble A</th>
<th>Gamble B</th>
<th>Which is better?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $45</td>
<td>.5 $100</td>
<td>vs. .5 $0</td>
</tr>
</tbody>
</table>

• Definition of expected utility
• Desirable properties of expected utility
  – rationality (consistency)
  – monotonicity (more is better)
  – measurability (any two gambles can be ranked)
Three Attitudes towards risk

• The main attitudes
  – risk neutral
  – risk averse
  – risk seeking
• An individual is said to be risk neutral if she is indifferent between accepting a gamble and a certain payment which equals the expected value of the gamble.
• An individual is said to be risk averse if she is strictly prefers a certain payment which equals the expected value of a gamble to the gamble itself.
• An individual is risk averse if the marginal gain from an extra dollar declines as she gets more dollars. (The utility function is concave.)
• Risk seeking?
• We typically expect individuals (firms?) to be risk neutral or risk averse.

Small Business, imperfect information

![Small Business, imperfect information diagram]
Small Business, risk-neutral player

\[ u(x) = x \]

\[
\begin{array}{c}
\text{Open} \\
50K \\
\text{Not Open} \\
10K \\
\text{Open} \\
0 \\
\text{Not Open} \\
10K \\
\end{array}
\]

Small Business, risk-averse player

\[ u(x) = x^{\frac{1}{2}} \]

\[
\begin{array}{c}
\text{Open} \\
447.2 \\
\text{Not Open} \\
100 \\
\text{Open} \\
0 \\
\text{Not Open} \\
100 \\
\end{array}
\]
Small Business, risk-seeking player

\[ u(x) = \frac{1}{2} x^2 \]