Physics 761 Homework #9 Fall '17

Dr. Drake

Due November 16

1. At very small spatial scales the MHD equations break down and the ion motion can often be neglected. The resulting equations are called the electron MHD equations. In this case the magnetic field moves with the electrons so

$$\mathbf{E} = -\frac{1}{c}\mathbf{v}_e \times \mathbf{B} - \frac{m}{e}\frac{d\mathbf{v}_e}{dt} \tag{1}$$

and the electron velocity $\mathbf{v}_e$ is related to the current $\mathbf{J} = -n_0 e \mathbf{v}_e$. The EMHD equations then take the form

$$\frac{\partial \mathbf{B}'}{\partial t} + \frac{1}{n_0 e}\nabla \times (\mathbf{J} \times \mathbf{B}') = 0; \tag{2}$$

$$\mathbf{B}' = \left(1 - \frac{c^2}{\omega_{pe}^2}\nabla^2\right)\mathbf{B}; \tag{3}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{4}$$

where $c/\omega_{pe}$ is the electron skin depth. Consider a uniform magnetic field $B_0$ in the $z$ direction. Calculate the dispersion relation for waves propagating along the magnetic field. Plot the frequency versus the wavevector $k$. What is the limiting frequency at very large values of $k$? At what value of $k$ does the frequency roll over to this value?

2. In a plasma with a strong magnetic field in the $z$ direction, the two-dimensional motion in the $x-y$ plane can be considered incompressible if the times scales of the motion are longer than the characteristic fast-wave propagation time. Thus, the velocity in the $x - y$ plane can be written as

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla\phi. \tag{5}$$

Show that the equation for the velocity potential $\phi$ is given by

$$\left(\frac{\partial}{\partial t} + \hat{\mathbf{z}} \times \nabla\phi \cdot \nabla\right)\nabla^2\phi = 0. \tag{6}$$

Hint: Show that the mass density $\rho$ is constant in time if it is initially spatially uniform. Operate with $\hat{\mathbf{z}} \cdot \nabla\times$ on the momentum equation. Show that if the components of the magnetic field in the $x - y$ plane are initially zero, they remain zero and therefore all of the magnetic tension forces are zero.

3. Consider a system with a flow $V_0$ that is in the positive $x$ direction with a magnitude $V_{00}$ for $y > 0$ and with flow with the same magnitude but opposite direction for $y < 0$. This equilibrium is unstable to the Kelvin-Helmholtz instability. Using the incompressible equations derived in the second problem and assuming that perturbations in the potential can be written as $Re(\hat{\phi}(y)exp(-i\omega t + ikx))$, show that the equation for $\hat{\phi}$ is given by

$$[(\omega - kV_0)\hat{\phi}']' + (k\hat{\phi}V_0')' - k^2(\omega - kV_0)\hat{\phi} = 0. \tag{7}$$

In the vicinity of $y = 0$ the equation simplifies and you can do an integration to show that

$$(\omega - kV_0)\hat{\phi}' + kV_0'\hat{\phi} = const. \tag{8}$$

This result means that $\hat{\phi}'$ undergoes a jump across $y = 0$. Calculate this jump. Then integrate Eq. (8) to show that

$$\hat{\phi} = C(\omega - kV_0), \tag{9}$$

where $C$ is a constant so that $\hat{\phi}$ also suffers a jump across $y = 0$. Calculate this jump. Away from $y = 0$ write the equation for $\hat{\phi}$ and solve it for $y > 0$ and $y < 0$. Use the jump conditions on $\hat{\phi}$ and $\hat{\phi}'$ to obtain an equation for the instability growth rate.