Physics 606 **Final Exam** **Spring 1997**
Dr. Drake

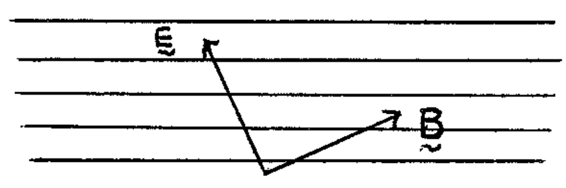
1. (70 points) The following are short answer questions which should not require extensive calculations.

   (a) Consider an infinite dielectric rod aligned with the z direction and coated with a uniform surface charge density σ. The rod oscillates around its axis with an angular displacement ϕ_0. Indicate the directions of the oscillatory components of the electric and magnetic fields. Sketch $\mathbf{E}$ in the $x-y$ plane and indicate the direction of $\mathbf{B}$. What are the relative magnitudes of the oscillatory components of $\mathbf{E}$ and $\mathbf{B}$ at large distances from the rod. How do they fall off with distance ρ from the rod in this region? How do the fields scale with the surface charge density σ and oscillation amplitude ϕ_0?

   (b) A plane electromagnetic wave of frequency ω is normally incident on a plane conductor which has a high conductivity σ. Estimate the fraction of energy which is dissipated in the conductor during the reflection of the wave. You can simply write down the answer without doing a calculation. What happens to the energy?

   (c) Consider a very long cylindrical rod which has a permanent magnetization M oriented along its axis. The cross-sectional area of the rod is A. What is the magnetic field inside of the rod? Two such rods are placed end to end with their magnetic moments in the same direction and separated by a short distance d with $d^2 << A$. What is the magnetic field inside of the gap. Estimate the force of attraction/repulsion between the two rods. Do they attract or repell each other?

   (d) An electromagnetic wave is incident on a grid of fine, very high conductivity wires as shown. The wavelength of the light is much greater than the spacing of the wires. What happens to the wave?

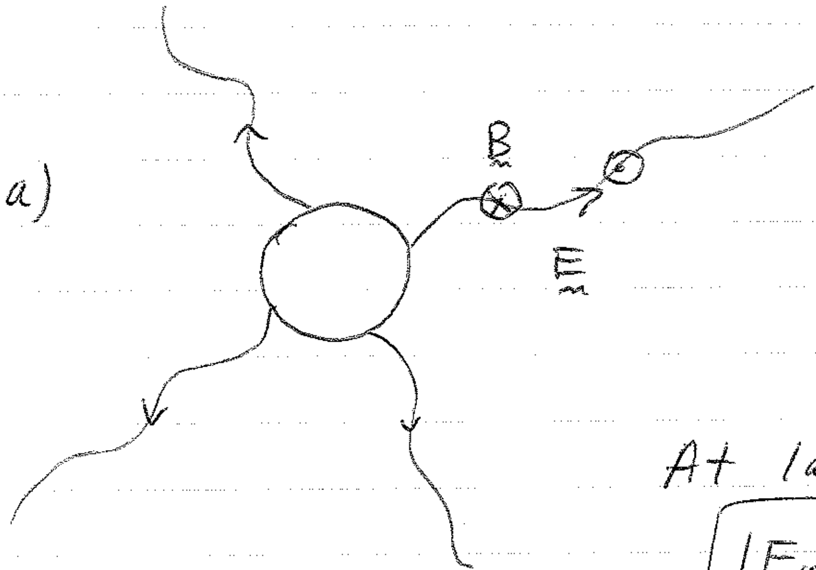


2. (60 points) Consider a waveguide consisting of a vacuum enclosed by a cylindrical conductor of radius "a" and infinite extent along z. Assume that the conductivity of the metal wall is infinite. The guide is excited with an antenna of frequency ω. In the following questions relate to the lowest order TM ($B_z = 0$) mode of the guide.

   (a) Sketch the electric and magnetic field lines for this mode. What are the nonzero components of $\mathbf{E}$ and $\mathbf{B}$?

Physics 606 Exam solutions

(1) a)



$$\left.\begin{array}{l} E_\varphi \neq 0 \\ B_z \neq 0 \end{array}\right\} \text{oscillatory components}$$

At large distances

$$\boxed{|E_\varphi| = |B_z|}$$

$\int d\varphi\,\rho\, S$ is indep of ρ with $S \sim E_\varphi B_z^*$

$$\Rightarrow \boxed{E_\varphi \sim B_z^* \sim \frac{1}{\rho^{1/2}}}$$

$\Rightarrow$ system is linear with the surface current produced by the oscillating ~~rods~~ rod so

$$\boxed{E_\varphi \sim B_z \sim \sigma \varphi_0}$$

b)

Fraction of energy absorbed

$$\sim k\delta = \frac{\omega}{c}\delta \ll 1$$

$$\text{with } \delta = \frac{c}{\sqrt{2\pi\sigma\omega}} = \text{skin depth}$$

c)

$$\underset{\sim}{H} = \underset{\sim}{B} - 4\pi \underset{\sim}{M}$$

$$\nabla \times \underset{\sim}{H} = 0 \implies H = -\nabla \varphi_m$$

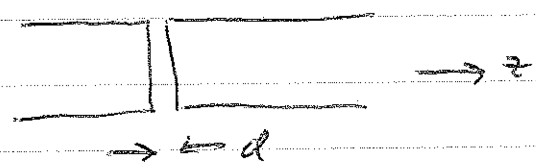
$$\nabla \cdot \underset{\sim}{B} = 0 = -\nabla^2 \varphi_m + 4\pi\, \nabla \cdot \underset{\sim}{M}$$

$\nabla \cdot \underset{\sim}{M} = 0$ away from ends

$$\Rightarrow \nabla^2 \Phi_m = 0 \Rightarrow \underset{\sim}{H} = 0$$

$$\Rightarrow \boxed{\underset{\sim}{B} = 4\pi \underset{\sim}{M}} \text{ inside}$$

B_z is continuous across gap



$$\Rightarrow B = 4\pi M \text{ inside of the gap.}$$

$$W_B \sim \frac{B^2}{8\pi} A d$$

$$F = -\frac{\partial W_B}{\partial d} = \boxed{-2\pi M^2 A} \text{ attraction}$$

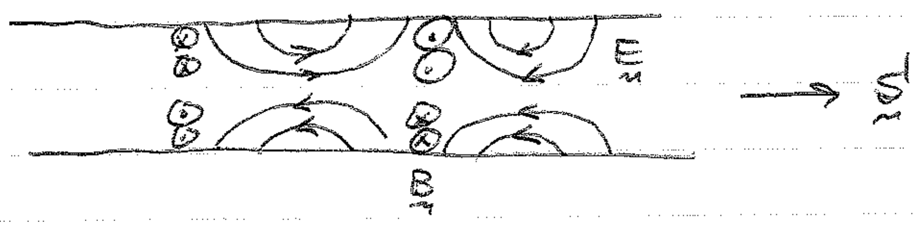
d) Split wave into components with $\underset{\sim}{E}$ along the wires and transverse to the wires.

Wave with $\underset{\sim}{E}$ transverse to the wires passes through with no ~~att~~ reflection

Wave with $\underset{\sim}{E}$ parallel to wires is reflected. Wave sees wires as a plane surface since the spacing is much shorter than the wavelength.

(2) TM mode $\Rightarrow$ $E_z \neq 0$

a)



$E_z, E_\rho \neq 0$ $\qquad$ $\frac{\partial}{\partial \varphi} = 0$ for lowest order mode

$B_\varphi \neq 0$

b)

$$\nabla \times \underset{\sim}{B} = \frac{1}{c} \frac{\partial}{\partial t} \underset{\sim}{E} \qquad \nabla \cdot \underset{\sim}{E} = 0$$

$$\nabla \times \underset{\sim}{E} + \frac{1}{c} \frac{\partial \underset{\sim}{B}}{\partial t} = 0 \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\underbrace{\nabla \times (\nabla \times E)}_{\nabla \cancel{\nabla \cdot \underset{\sim}{E}}^{\,0} - \nabla^2 \underset{\sim}{E}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \underset{\sim}{E} = 0$$

$$E_z \sim e^{ikz} e^{-i\omega t}$$

$$\nabla^2 E_z + \frac{\omega^2}{c^2} E_z = 0$$

$$\boxed{\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - k^2 + \frac{\omega^2}{c^2} \right) E_z = 0} \qquad \boxed{E_z(\rho = a) = 0}$$

c) $$E_z(\rho) = E_0 J_0(\gamma \rho)$$

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2$$

$$\Rightarrow J_0(\gamma a) = 0$$

$$\left(\frac{\omega^2}{c^2} - k^2\right) a^2 = X_{01}^2$$

$$\Rightarrow \boxed{\frac{\omega^2}{c^2} = k^2 + \frac{X_{01}^2}{a^2}}$$

Energy propagates at the group velocity

$$v_g = \frac{d\omega}{dk}$$

$$\frac{\cancel{2}\omega v_g}{c^2} = \cancel{2}k$$

$$v_g = \frac{kc^2}{\omega} = \frac{\left(\frac{\omega^2}{c^2} - \frac{X_{01}^2}{a^2}\right)^{\frac{1}{2}} c^2}{\omega}$$

$$\boxed{v_g = c\left(1 - \frac{X_{01}^2 c^2}{\omega^2 a^2}\right)^{\frac{1}{2}}}$$

$v_g = c$ for ω large

$\Rightarrow$ free space wavelength short compared to radius of the guide

$\Rightarrow$ wave doesn't really see the guide so propagates at c

$\Rightarrow$ does no propagate at low frequency

$\Rightarrow$ cutoff frequency $\boxed{\omega_c = \frac{X_{01} c}{a}}$

3 a) $\underset{\sim}{E} = E_0 \hat{y} \cos(kx - \omega t)$

$\underset{\sim}{B} = E_0 \hat{z} \cos(kx - \omega t)$ $\qquad \omega = kc$

Transform $kx - \omega t$ to S' frame

$$x = \gamma(x' - \beta c t')$$

$$ct = \gamma(ct' - \beta x')$$

$$kx - \omega t = k\gamma(x' - \beta c t') - \omega \gamma\left(t' - \frac{\beta}{c} x'\right)$$

$$= \gamma x'\left(k + \beta \frac{\omega}{c}\right) - t'\gamma(\omega + \beta k c)$$

$$\boxed{\begin{aligned} k' &= \gamma\left(k + \beta \frac{\omega}{c}\right) = k\gamma(1+\beta) \\ \frac{\omega'}{c} &= \gamma\left(\frac{\omega}{c} + \beta k\right) = \gamma \frac{\omega}{c}(1+\beta) \end{aligned}}$$

$\left(\frac{\omega}{c}, \underset{\sim}{k}\right)$ form a 4-vector

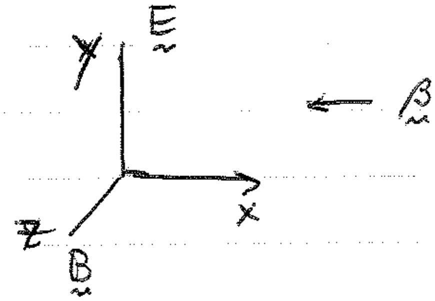
phase $\omega t - \underset{\sim}{k} \cdot \underset{\sim}{x} = k \cdot X$

$k \cdot X$ is a Lorentz scalar

b)

$$\underset{\sim}{E}'_\perp = \gamma\left(\underset{\sim}{E}_\perp + \underset{\sim}{\beta} \times \underset{\sim}{B}_\perp\right)$$

$$\underset{\sim}{B}'_\perp = \gamma\left(\underset{\sim}{B}_\perp - \underset{\sim}{\beta} \times \underset{\sim}{E}_\perp\right)$$

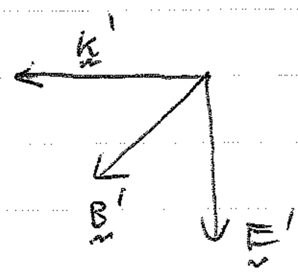


$$E_y' = \gamma\left(E_y + \beta B_z\right) = \boxed{\gamma E_y (1+\beta)}$$

$$\boxed{B_z' = \gamma E_y (1+\beta)}$$

fields have increased

c) Reflected wave



$$\underset{\sim}{E}' = -E_y' \cos(k'x' + \omega' t')\,\hat{y} = -\gamma(1+\beta) E_y \cos(\ )\,\hat{y}$$

$$\underset{\sim}{B}' = B_z' \cos(k'x' + \omega' t')\,\hat{z} = \gamma(1+\beta) E_y \cos(\ )\,\hat{z}$$

Transform back to S

$$\underset{\sim}{E}_\perp = \gamma\left(\underset{\sim}{E}'_\perp - \underset{\sim}{\beta} \times \underset{\sim}{B}'\right)$$

$$E_{y_r} = \gamma\left(E_y' - \beta B_z'\right)$$

$$E_{y_r} = -\gamma^2 (1+\beta)^2 E_y \cos(\ ) = -\frac{1+\beta}{1-\beta} E_y \cos(\ )$$

$$B_{z_r} = \gamma^2 (1+\beta)^2 E_y \cos(\ ) = \frac{1+\beta}{1-\beta} E_y \cos(\ )$$

$k'x' + \omega' t'$

$\Rightarrow$ transform back to S' frame

$$\begin{pmatrix} \frac{\omega_r}{c} \\ k_r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\omega'}{c} \\ k' \\ 0 \\ 0 \end{pmatrix}$$

$$\omega_r = -\gamma(1+\beta)\omega' \qquad \omega_r = -\gamma^2(1+\beta)^2\omega$$

$$k_r = k'\gamma(1+\beta) \qquad k_r = +\gamma^2(1+\beta)^2 k$$

$$\boxed{\begin{aligned} \omega_r &= -\frac{1+\beta}{1-\beta}\,\omega \\ k_r &= \frac{1+\beta}{1-\beta}\,k \end{aligned}}$$

$$k'x' + \omega' t' \Rightarrow \boxed{k_r x - \omega_r t}$$

$\Rightarrow$ frequency and k vector increase.

d) Momentum density of incident wave.

$$P_I = \frac{1}{8\pi c} E B^* = \frac{E_0^2}{8\pi c}$$

$$F_I = \frac{\text{force}}{\text{area}} \text{ from incident wave}$$

$$= \frac{P_I (c\Delta t + v\Delta t)}{\Delta t}$$

$$F_I = \frac{E_0^2}{8\pi}(1+\beta)$$

$$P_R = \frac{E_r B_r^*}{8\pi c} = -\frac{E_0^2}{8\pi c}\left(\frac{1+\beta}{1-\beta}\right)^2$$

$$F_R = \frac{E_0^2}{8\pi c}\left(\frac{1+\beta}{1-\beta}\right)^2 \frac{(c\Delta t - v\Delta t)}{\Delta t}$$

$$= \frac{E_0^2}{8\pi}\frac{(1+\beta)^2}{1-\beta}$$

$$F_{TOT} = \frac{E_0^2}{8\pi}\left(1+\beta + \frac{(1+\beta)^2}{1-\beta}\right)$$

$$= 2\frac{E_0^2}{8\pi}\left(\frac{1+\beta}{1-\beta}\right)$$