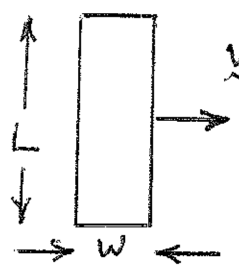
Physics 606 Homework #8 Spring '12
Dr. Drake

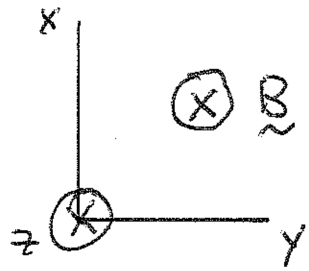
1. Jackson 5.27

2. Jackson 6.11
Hint: The solar wind velocity is around $400km/s$ and the proton number density is around $10/cm^3$.

3. Jackson 6.14

4. Consider a rectangular loop of wire as shown with $L \gg w$ incident on a magnetized half space, $\mathbf{B} = B_0 H(y)\hat{z}$, where H is the Heaviside function. The wire





has an initial velocity $\mathbf{v} = v_0\hat{y}$ with $v_0 \ll c$, a mass per unit length λ and infinite conductivity. The wire making up the loop has a circular cross section with a radius $a \ll w$. Initially the wire does not carry any current. Neglect end effects in the following.

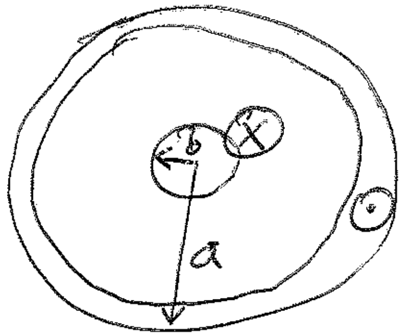
   (a) Under the assumption that the plane of the wire is constrained to remain perpendicular to $\hat{z}$ calculate the total current flowing around the wire as the loop enters the magnetic field. Express you answer in terms of the distance y the right side of the wire has penetrated the magnetic field. Be sure to indicate the direction of the current.

   (b) Calculate the force per unit length acting on the wire and write down an equation of motion for the wire.

   (c) Solve for the motion of the wire for all possible values of v_0.
   Hint: There are two possible final states. What is the current in the wire in these two final states?

Hwk #8 Solutions

5.27



Calculate the inductance per unit length for currents going in the inner conductor and out the outer.

$\Rightarrow$ calculate the stored energy

$$\frac{dW_B}{dl} = \frac{1}{2}\frac{dL}{dl} I^2$$

$r > b$

$$\nabla \times \underset{\sim}{B} = \mu_0 J$$

$$B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

$r < b$

$$B 2\pi r = \mu_0 \frac{\pi r^2}{\pi b^2} I$$

$$B = \mu_0 \frac{r}{b^2} I \frac{1}{2\pi}$$

$$\frac{dW_B}{dl} = \int_0^b 2\pi r\, dr \left(\frac{\mu_0 r I}{2\pi b^2}\right)^2 \frac{1}{2\mu_0}$$

$$+ \int_b^a 2\pi r\, dr \left(\frac{\mu_0 I}{2\pi r}\right)^2 \frac{1}{2\mu_0}$$

$$= \cancel{\frac{2\pi}{2\mu_0}} \frac{\mu_0^{\cancel{2}} I^2}{4\pi^{\cancel{2}} \cancel{b^4}} \frac{\cancel{b^4}}{4} + \cancel{\frac{2\pi}{2\mu_0}} \frac{\mu_0^{\cancel{2}} I^2}{4\pi^{\cancel{2}}} \ln\left(\frac{a}{b}\right)$$

$$\frac{1}{2}\frac{dL}{dl} I^2 = \frac{\mu_0 I^2}{4\pi}\left[\frac{1}{4} + \ln\frac{a}{b}\right]$$

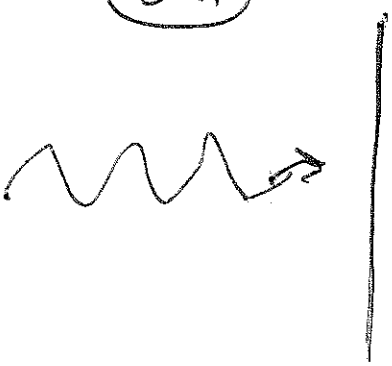
2

$$\boxed{\frac{dL}{dl} = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln\frac{a}{b}\right)}$$

For thin hollow tube on inside

$$\boxed{\frac{dL}{dl} = \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right)}$$

8.11



a) Force is momentum per unit time deposited.

$$\underline{P}_{field} = \frac{1}{c^2}\,\underline{E}\times\underline{H}$$

In time Δt, momentum deposited is

$$A\,\cancel{c}\,\Delta t\ \frac{1}{c^{\cancel{2}}}\frac{EB}{\mu_0} = F\Delta t = P_{rad}\,A\Delta t$$

$$P_{rad} = \frac{EB}{c\mu_0} = \frac{B^2}{\mu_0} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

$$= u = \text{energy density}$$

b)

$$a\,\frac{dm}{dA} = u \qquad uc = 1.4\,\frac{\text{kW}}{\text{m}^2}$$

$$a = \frac{1.4\times10^{3}\ \cancel{W}}{10^{-3}\ \cancel{g}\ \cancel{kg}}\ \frac{\cancel{Nm}}{s}\ \frac{s}{3\times10^{8}\,\cancel{m}}\ \frac{\cancel{kg}\,\cancel{m}}{s^{\cancel{2}}} = 4.67\times10^{-3}\ m/s^2$$

From the solar wind

$$10^{-3}\,\frac{kg}{m^2}\,a = 1.67\times10^{-27}\,\frac{kg\ 10}{(10^{-2}\,m)^3}\left(\frac{400\,km}{s}\right)^2$$

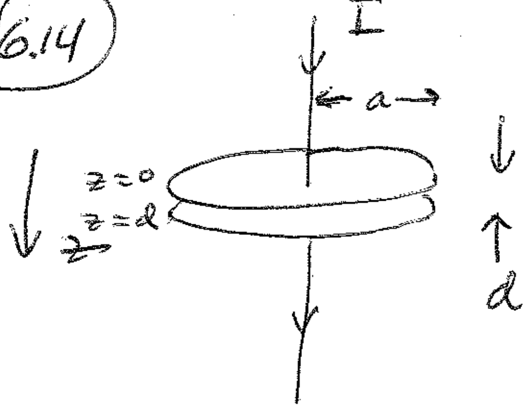
$$10^{-3}\,\cancel{\frac{kg}{m^2}}\,a = \frac{1.67\times10^{-27}(10)}{10^{-6}\,\cancel{m^3}}\,\cancel{kg}\,16\times10^4\,\frac{\cancel{km}\,10^6\,m^2}{s^2}$$

$$a = 26.7\times10^{-7}\,\frac{m}{s^2}$$

$$= 2.67\times10^{6}\,\frac{m}{s^2}$$

$\Rightarrow$ much less than radiation pressure

6.14



$I = I_0 \cos(\omega t)$

$d \ll a \qquad Q_0 = \dfrac{\sin \omega t \, I_0}{\omega}$

expansion parameter

$\delta \equiv \dfrac{\omega a}{c}$

$\Rightarrow$ light propagation time is short compared with frequency

a) Maxwell Eqns $\qquad \epsilon = \epsilon_0, \ \mu = \mu_0$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

$$\nabla \cdot \underset{\sim}{E} = \rho / \epsilon_0$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0 \quad \Rightarrow \text{conductor inner surface}$$

$\Rightarrow$ treat $\frac{\omega a}{c}$ as the small parameter and solve the equations order by order in this small parameter.

Lowest order (zero order)

$$\nabla \times \underset{\sim}{B}_0 = \mu_0 \underset{\sim}{J}_0$$

$$\nabla \times \underset{\sim}{E}_0 = 0$$

$$\nabla \cdot \underset{\sim}{E}_0 = \rho_0 / \epsilon_0$$

$$\nabla \cdot \underset{\sim}{B}_0 = 0 \qquad \nabla \cdot \underset{\sim}{J}_0 = 0$$

J_0 is current on ~~cond~~ inner surface of the capacitor

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho J_{0\rho} = 0$$

$$\rho J_{0\rho} = \text{const} \Rightarrow \boxed{J_{0\rho} = 0}$$

$$\Rightarrow \nabla\times B_0 = 0 \text{ and } \nabla\cdot B_0 = 0$$

$$\Rightarrow \boxed{B_0 = 0}$$

$$\nabla\times E_0 = 0 \Rightarrow E_0 = -\nabla\varphi_0$$

$$\hat{n}\times E_0 = 0 \Rightarrow \frac{\partial\varphi_0}{\partial\rho} = 0$$

$\Rightarrow$ ideal conductor

$$\Rightarrow \frac{\partial^2}{\partial z^2}\varphi_0 = -\frac{\rho_0}{\epsilon_0} \Rightarrow \rho_0 = \rho_0(z) = \sigma_0\,\delta(z)$$

$$\sigma_0 = \frac{Q_0}{\pi a^2} = \frac{I_0}{\pi a^2}\frac{\sin\omega t}{\omega}$$

$$\boxed{E_{z0} = \frac{\sigma_0}{\epsilon_0} = \frac{I_0}{\epsilon_0\pi a^2}\frac{\sin\omega t}{\omega}}$$

<u>First order</u>

$$\nabla\times B_1 = \cancel{\mu_0 J_1} + \mu_0\epsilon_0\frac{\partial}{\partial t}E_0$$

$$\left.\begin{aligned} &\nabla\times E_1 + \cancelto{0}{\frac{\partial}{\partial t}B_0} = 0 \\ &\nabla\cdot E_1 = \rho_1/\epsilon_0 = 0 \end{aligned}\right\} \quad E_1 = 0$$

$$\nabla\cdot B_1 = 0$$

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot \underset{\sim}{J}_1 = 0$$

<u>in vacuum</u> $\quad (\nabla \times \underset{\sim}{B}_1)_z = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_{0z}$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho B_{1\varphi} = \mu_0 \cancel{\epsilon_0} \frac{\cancel{\omega} I_0}{\pi a^2} \frac{\cos\omega t}{\cancel{\omega \epsilon_0}}$$

$$\boxed{B_{1\varphi} = \mu_0 \frac{I_0}{\pi a^2} \frac{\rho}{2} \cos\omega t}$$

note: $B_{1\varphi} = 0$ inside cap. plate

$\Rightarrow$ surface current $J_{1\rho} \neq 0$

At the surface

$$(\nabla \times \underset{\sim}{B}_{1})_\rho = \mu_0 J_{1\rho}$$

$$\underbrace{-\frac{\partial}{\partial z} B_{1\varphi}}_{-B_{1\varphi}\delta(z)} = \mu_0 J_{1\rho}$$

$$J_{1\rho} = -\frac{1}{\mu_0} B_{1\varphi}\, \delta(z)$$

check continuity of charge

$$\frac{\partial \rho_0}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho J_{1\rho} = 0$$

$$\frac{I_0}{\pi a^2}\cancel{\delta(z)} \cos\omega t + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(-\frac{\rho}{\mu_0} B_{1\varphi} \cancel{\delta(z)}\right) = 0$$

$$\cancel{\frac{I_0}{\pi a^2}\cos\omega t} - \cancel{\frac{\mu_0 I_0}{\pi a^2}\cos\omega t \frac{1}{\mu_0}} = 0 \qquad \text{OK}$$

<u>second order</u>

$$\nabla \times \underset{\sim}{B}_2 = \mu_0 J_2 + \cancel{\mu_0 \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}_1}^{\,0}$$

$$\nabla \times \underset{\sim}{E}_2 + \frac{\partial}{\partial t} \underset{\sim}{B}_1 = 0$$

$$\nabla \cdot \underset{\sim}{E}_2 = \rho_2/\epsilon_0$$

$$\nabla \cdot \underset{\sim}{B}_2 = 0$$

$$\cancel{\frac{\partial \rho_1}{\partial t}}^{\,0} + \nabla \cdot \underset{\sim}{J}_2 = 0 \quad \Longrightarrow \underset{\sim}{J}_2 = 0$$

$$\Longrightarrow \left.\begin{array}{l} \nabla \times \underset{\sim}{B}_2 = 0 \\ \nabla \cdot \underset{\sim}{B}_2 = 0 \end{array}\right\} \underset{\sim}{B}_2 = 0$$

since $\hat{z} \times \underset{\sim}{E}_2 = 0$ at $z = 0, d$

$$\Longrightarrow E_{2\rho},\ E_{2\varphi} = 0$$

$$\Longrightarrow E_{2z} \neq 0$$

$$-\frac{\partial}{\partial \rho} E_{2z} + \frac{\partial}{\partial t} B_{1\varphi} = 0$$

$$-\frac{\partial}{\partial \rho} E_{2z} + \left(-\frac{\mu_0 I_0}{\pi a^2} \frac{\rho}{2} \omega \sin \omega t\right) = 0$$

$$E_{2z} = -\frac{\mu_0 I_0}{\pi a^2} \omega \frac{\rho^2}{4} \sin \omega t + E_{2z}(0)$$

$$E_{2z} = -E_{z0} \frac{\omega^2}{c^2} \rho^2 \frac{1}{4} + E_{2z}(0)$$

$$= \frac{\sigma_2}{\epsilon_0}$$

but $\int da\, \sigma_2 = 0 = E_{2z}(0) \pi a^2 - E_{z0} \frac{\omega^2}{c^2} \frac{2\pi a^4}{16}$

$$E_{2z}(0) = E_{z0}\,\frac{\omega^2}{c^2}\,a^2\,\frac{1}{8}$$

$$\boxed{E_{2z}(\rho) = E_{z0}\,\frac{\omega^2}{c^2}\,\frac{1}{8}\left(a^2 - 2\rho^2\right)}$$

b) time averaged stored energy

$$W_E = \frac{1}{2}\epsilon_0 \int d\underline{x}\,\langle E^2\rangle$$

$$= \frac{1}{2}\epsilon_0\,\frac{I_0^2}{\epsilon_0^2\pi^2 a^4}\,\frac{1}{\omega^2}\,\frac{1}{2}\,\pi a^2 d$$

$$= \frac{I_0^2 d}{4\pi\epsilon_0\omega^2 a^2}$$

$$W_B = \frac{1}{2\mu_0}\int d\underline{x}\,\langle B^2\rangle$$

$$= \frac{1}{2\cancel{\mu_0}}\,\frac{\mu_0^{\cancel{2}} I_0^2}{\pi^{\cancel{2}} a^4}\,\frac{1}{4}\,\frac{1}{\cancel{2}}\,d\,(2\pi)\underbrace{\int_0^a d\rho\,\rho\,\rho^2}_{a^4/4}$$

$$\boxed{W_B = \frac{\mu_0 I_0^2 d}{32\pi}}$$

c) capacitance $\frac{1}{2}C\langle V^2\rangle = W_E$

$$\frac{1}{2}C\,\frac{d^2\,\cancel{I_0^2}}{\epsilon_0^2\pi^2 a^4}\,\frac{1}{2}\,\cancel{\frac{1}{\omega^2}} = \frac{\cancel{I_0^2}\,d}{4\pi\epsilon_0\cancel{\omega^2}\,a^2}$$

$$\boxed{C = \frac{\epsilon_0\pi a^2}{d}}$$

$$\frac{1}{2} L I^2 = w_B$$

$$\frac{1}{2} L \frac{\cancel{I_0^2}}{2} = \frac{\mu_0 \cancel{I_0^2} d}{32\pi}$$

$$\boxed{L = \frac{\mu_0 d}{8\pi}}$$

<u>resonance frequency</u>

$$\frac{1}{2} L I^2 + \frac{1}{2} C V^2 = \text{const}$$

$$L I \dot{I} + \cancel{2} C V \dot{V} = 0$$

$$V = \underline{Ed} = \frac{Q}{A\epsilon_0} d$$

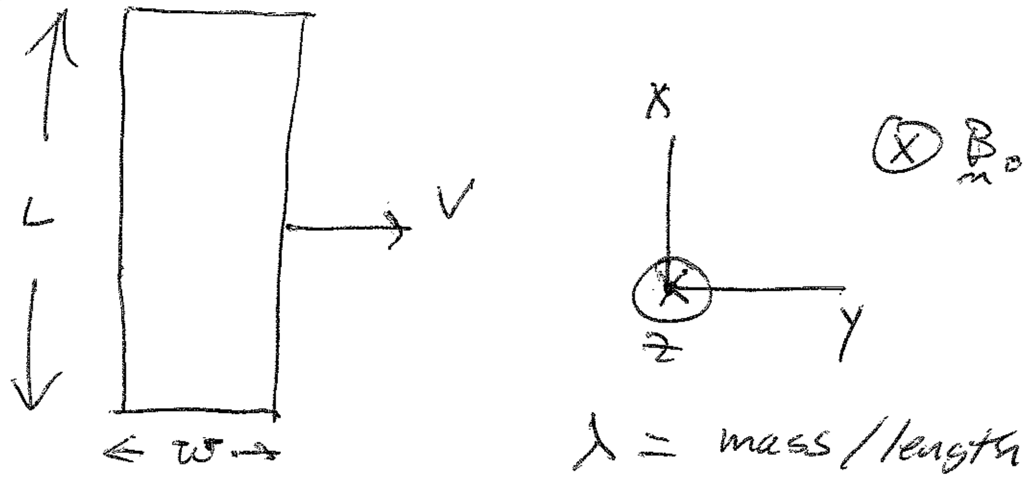
$$L \cancel{I} \ddot{Q} + C \frac{Q}{A\epsilon_0} d \frac{\cancel{I} d}{A\epsilon_0} = 0$$

$$\ddot{Q} + \frac{C d^2}{A^2 \epsilon_0^2 L} Q = 0$$

$$\omega_{res} = \sqrt{\frac{C}{L}} \frac{d}{A\epsilon_0}$$

$$= \sqrt{\frac{\cancel{\pi}\epsilon_0 \cancel{a^2} 8\pi}{\cancel{d}\mu_0 \cancel{d}}} \frac{\cancel{d}}{\cancel{\pi} a^2 \epsilon_0} = \boxed{\frac{c}{a} 2\sqrt{2}}$$

4



λ = mass / length

a) Since the wire is a perfect conductor, the Emf must remain zero so that I in the loop is finite. The magnetic flux cutting through the loop by Bo must be balanced by that due to the current I.

The ~~xxxx~~ magnetic field from a wire is given by

$$B = \frac{\mu_o I}{2\pi r}$$

Flux from one side

$$\Psi = L \int_a^w dr \, \frac{\mu_o I}{2\pi r} = L \frac{\mu_o I}{2\pi} \ln\left(\frac{w}{a}\right)$$

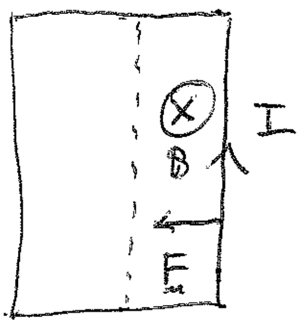
total flux is twice this. Note current is on wire surface so no flux inside wire matching fluxes

$$B_0 \cancel{L} y = \frac{\cancel{2L} \mu_0 I}{\cancel{2}\pi} \ln\left(\frac{w}{a}\right)$$

where y is the distance the loop has penetrated the flux. The current is counter clockwise

$$\boxed{I = \frac{\pi B_0 y}{\mu_0 \ln\left(\frac{w}{a}\right)}}$$

b) $\underline{F} = I\, \underline{L} \times \underline{B}$ with $\underline{B}$ only the external field B_0.



$\underline{F}$ is to the left

$$F = I L B_0$$

$$\frac{dF}{dL} = \frac{\pi B_0^2 \cancel{L}}{\mu_0 \ln\left(\frac{w}{a}\right)} y$$

$$\boxed{\lambda \ddot{y} = -\frac{\pi B_0^2}{\mu_0 \ln\left(\frac{w}{a}\right)} y}$$

c) Let $\Omega^2 = \dfrac{\pi B_0^2}{\lambda \mu_0 \ln \frac{w}{a}}$

Let the loop enter B_0 at $t=0$

$$y = y_0 \sin \Omega t$$

$$\dot{y} = v_y = \Omega y_0 \cos \Omega t$$

$\dot{y}$ $v_y = V_0$ at $t=0$

$$\Rightarrow \quad y_0 = \frac{V_0}{\Omega}$$

$$\boxed{\begin{aligned} y &= \frac{V_0}{\Omega} \sin \Omega t \\ V_y &= V_0 \cos \Omega t \end{aligned}}$$

solution valid until the loop completely enters the field or until it returns to $y=0$

<u>low velocity</u> $\quad \frac{V_0}{\Omega} < w$

At $\Omega t = \pi$ the loop will return to $y=0$ and exit the field with a velocity $-V_0$

high velocity $\quad \frac{v_0}{\Omega} > \omega$

At
$$\frac{v_0}{\Omega} \sin(\Omega \tau) = \omega$$

the loop will completely enter the loop and the loop will ~~move~~ remain at a constant velocity given by

$$v_y = v_0 \left(1 - \sin^2 \Omega \tau\right)^{\frac{1}{2}}$$

$$= v_0 \left(1 - \frac{\omega^2}{v_0^2} \Omega^2\right)^{\frac{1}{2}}$$

$$\boxed{v_{y\,\text{final}} = \left(v_0^2 - \omega^2 \Omega^2\right)^{\frac{1}{2}}}$$