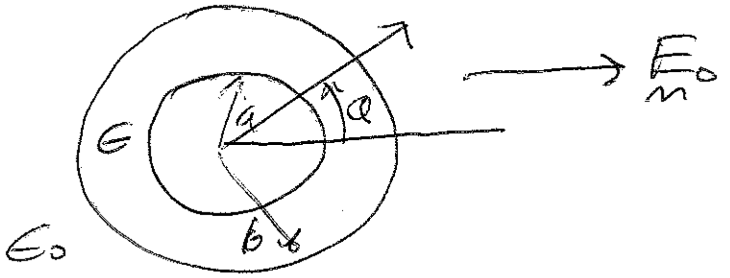
Hwk #5 Solutions

(4.8) Long, cylindrical shell of dielectric constant ϵ



$\underline{r > b}$

$$\Phi = \sum_n a_n \left(\frac{b}{\rho}\right)^n \cos n\varphi - E_0 \rho \cos\varphi$$

$\Rightarrow$ even in φ

$\Rightarrow$ symmetry

$\underline{a < r < b}$

$$\Phi = \sum_n \left[b_n \left(\frac{\rho}{b}\right)^n + c_n \left(\frac{b}{\rho}\right)^n \right] \cos n\varphi$$

$\underline{0 < r < a}$

$$\Phi = \sum_n d_n \left(\frac{\rho}{a}\right)^n \left[b_n \left(\frac{a}{b}\right)^n + c_n \left(\frac{b}{a}\right)^n \right] \cos n\varphi$$

For $n \neq 1$ have 2 BCs at "a" and "b"

$\Rightarrow$ over constrained

$\Rightarrow a_n, b_n, c_n, d_n = 0$

$\underline{\text{For } n = 1}$

radial $\underset{\sim}{D}$

at b: ① $\epsilon_0\left(-a_1 \frac{1}{b} - E_0\right) = \left(\frac{b_1}{b} - c_1 \frac{1}{b}\right)\epsilon$

at a: ② $d_1 \frac{E_0}{a}\left[b_1 \frac{a}{b} + c_1 \frac{b}{a} \right] = \left(b_1 \frac{1}{b} - c_1 \frac{b}{a^2}\right)\epsilon$

Azimuthal E_ϕ

at b: (3) $a_1 - E_0 b = b_1 + c_1$

at a: $b_1\left(\frac{a}{b}\right) + c_1\left(\frac{b}{a}\right) = d_1\left[b_1\left(\frac{a}{b}\right) + c_1\frac{b}{a}\right]$

$$\Rightarrow d_1 = 1$$

Eqn (2) becomes

$$b_1\left(\frac{\epsilon_0}{b} - \frac{\epsilon}{b}\right) = -c_1\frac{b}{a^2}(\epsilon + \epsilon_0)$$

$$\frac{b_1}{\cancel{b}}(\epsilon - \epsilon_0) = c_1\frac{b^2}{a^2}(\epsilon + \epsilon_0)$$

$$c_1 = \frac{a^2}{b^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\, b_1$$

Eq (3) becomes

(4) $$a_1 = E_0 b + b_1\left[1 + \frac{a^2}{b^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right]$$

Eq. (1) becomes

(5) $$a_1 + E_0 b = -\frac{\epsilon}{\epsilon_0}\left(1 + \frac{a^2}{b^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) b_1$$

subtract (5) from (4). Let $h \equiv \frac{a^2}{b^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}$

$$-2E_0 b = b_1\left[1 + h + \frac{\epsilon}{\epsilon_0}(1 + h)\right]$$

$$\cancel{-2\epsilon_0 E_0 b = b_1\left[\epsilon + \epsilon_0 + h(\epsilon + \epsilon_0)\right]}$$

$$\cancel{= b_1\left[\left(1 + \frac{a^2}{b^2}\right)(\epsilon - \epsilon_0)\right]}$$

$$b_1 = -\,2E_0 b \,\frac{\epsilon_0}{\epsilon+\epsilon_0}\,\frac{1}{1+\frac{a^2}{b^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}}$$

~~note that~~ $$c_1 = \frac{a^2}{b^2}\,\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\, b_1$$

Add (4) and (5)

$$2a_1 = b_1 (1+h)\left(1-\frac{\epsilon}{\epsilon_0}\right)$$

$$a_1 = +\frac{1}{\cancel{2}}\left(\frac{\epsilon-\epsilon_0}{\epsilon_0}\right)\left(+\frac{\cancel{2}E_0 b\,\epsilon_0}{\epsilon+\epsilon_0}\right)$$

$$a_1 = \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\,E_0 b$$

note for $\epsilon=\epsilon_0 \Rightarrow b_1 = -E_0 b$

$c_1 = a_1 = 0$

$\Rightarrow$ uniform field

$\underline{r>b}$

$$\Phi = a_1 \frac{b}{\rho}\cos\varphi - E_0 \rho \cos\varphi$$

$\underline{a<r<b}$

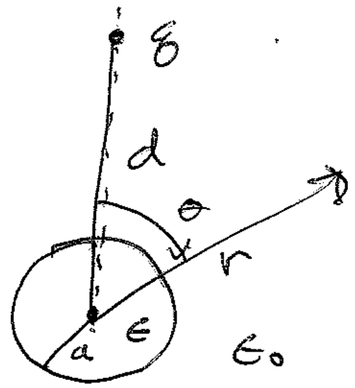
$$\Phi = \left(b_1 \frac{\rho}{b} + c_1 \frac{b}{\rho}\right)\cos\varphi$$

$\underline{r<a}$

$$\Phi = \frac{\rho}{a}\left[b_1 \frac{a}{b} + c_1 \frac{b}{a}\right]\cos\varphi$$

4.9

a)



You could solve this by expanding in spherical harmonics for $r > d$, $a < r < d$ and $r < a$ and doing the jump conditions at $r = d$ and $r = a$ but since you know the solution for q in the absense of the sphere, it is easier to write the potential in terms of that due to q, Φ_q, and that due to the sphere, Φ_ϵ. Then you only ~~had~~ have to do matching at $r = a$.

<u>charge q</u>

$$\Phi_q = \frac{q}{4\pi\epsilon_0} \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$\Rightarrow$ note $\frac{\partial}{\partial\varphi} = 0$

$r_<$ = smaller of r, d

$r_>$ = larger of r, d

<u>dielectric ϵ</u>

$r > a$ $$\Phi_\epsilon = \frac{q}{4\pi\epsilon_0} \sum_\ell a_\ell \left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

$r < a$ $$\Phi_\epsilon = \frac{q}{4\pi\epsilon_0} \sum_\ell b_\ell \left(\frac{r}{a}\right)^{\ell} P_\ell(\cos\theta)$$

BCs at $r = a$

continuity of E_θ $\Rightarrow$ Φ_q already continuous

$$-\frac{q}{4\pi\epsilon_0} \sum_\ell a_\ell \, P_\ell^1(\cos\theta)\sin\theta$$

$$= -\frac{q}{4\pi\epsilon_0} \sum_\ell b_\ell \, P_\ell^1(\cos\theta)\sin\theta$$

note: $\left[\frac{d}{d\cos\theta} P_\ell(\cos\theta)\right]\sin\theta \sim P_\ell^1(\cos\theta)$

$\Rightarrow$ orthogonal

$\Rightarrow$ can reduce series to single P_ℓ

$$\boxed{a_\ell = b_\ell}$$

continuity of D_r : $\Phi_q = \frac{q}{4\pi\epsilon_0} \sum_\ell \frac{r^\ell}{d^{\ell+1}} P_\ell$

$\Rightarrow$ project to single ℓ $\Rightarrow$ P_ℓ orthogonal

$$\epsilon_0 \left[\ell \frac{a^{\ell-1}}{d^{\ell+1}} - a_\ell (\ell+1) \frac{1}{a} \right]$$

$$= \epsilon \left[\ell \frac{a^{\ell-1}}{d^{\ell+1}} + \ell \frac{b_\ell}{a} \right]$$

$$a_\ell \left(\ell\epsilon + (\ell+1)\epsilon_0 \right) = -\frac{\ell a^\ell}{d^{\ell+1}} (\epsilon - \epsilon_0)$$

$$b_\ell = a_\ell = -\frac{a^\ell}{d^{\ell+1}} \frac{\ell(\epsilon-\epsilon_0)}{\ell\epsilon+(\ell+1)\epsilon_0}$$

b) near center of sphere

$$\Phi = \frac{q}{4\pi\epsilon_0} \sum_\ell \left[\frac{r^\ell}{d^{\ell+1}} + b_\ell \frac{r^\ell}{a^\ell} \right] P_\ell(\cos\theta)$$

$\Rightarrow$ only $\ell=1$ contributes as $r \to 0$

$$\Phi = \frac{q}{4\pi\epsilon_0} z \left[\frac{1}{d^2} + \frac{b_1}{a} \right]$$

$$b_1 = -\frac{a}{d^2} \frac{(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}$$

$$\Phi = \frac{q}{4\pi\epsilon_0} z \left[\frac{1}{d^2} - \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} \right]$$

$$E_z = -\frac{q}{4\pi\epsilon_0} \frac{1}{d^2} \left[1 - \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} \right]$$

$$= -\frac{q}{4\pi\epsilon_0 d^2} \frac{3\epsilon_0}{\epsilon+2\epsilon_0}$$

c) As $\epsilon \to \infty$

$E_z \to 0$

4.13

Have long cylindrical capacitor of inner radius "a" and outer radius "b". Applied voltage V. How far up will a fluid of dielectric constant ϵ move?

First consider the energy per unit length in capacitor with charge/length λ

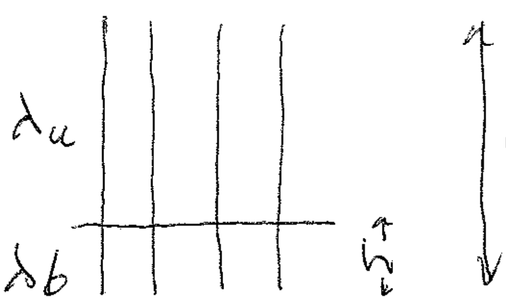
$$E = \frac{1}{2\pi\epsilon} \lambda \frac{1}{r}$$

$$w = \frac{1}{2}\epsilon \int d\underset{\sim}{x}\, E^2$$

$$= \frac{1}{2}\epsilon \int_a^b dr\, 2\pi r \frac{\lambda^2}{4\pi^2\epsilon^2 r^2}$$

$$= \frac{\lambda^2}{4\pi\epsilon} \ln\left(\frac{b}{a}\right)$$

Now consider a capacitor of length L and charge Q. What is the energy with a dielectric ϵ with the dielectric in a distance h.



$\Rightarrow$ the free charge moves so E is a constant along the entire capacitor

λ_a = charge/length in vacuum

λ_b = " " in dielectric

$$\lambda_a(L-h) + \lambda_b h = Q$$

$$\frac{\lambda_a}{\epsilon_0} = \frac{\lambda_b}{\epsilon} \Rightarrow E \text{ const.}$$

$$\lambda_b = \frac{\epsilon}{\epsilon_0} \lambda_u$$

$$\lambda_u \left[L - h + \frac{\epsilon}{\epsilon_0} h \right] = Q$$

$$W_{tot} = \frac{\ln\left(\frac{b}{a}\right)}{4\pi} \left[\frac{\lambda_u^2}{\epsilon_0}(L-h) + \frac{\lambda_b^2}{\epsilon} h \right]$$

$$= \frac{\ln\left(\frac{b}{a}\right)}{4\pi\epsilon_0} \left[L - h + \frac{\epsilon}{\epsilon_0} h \right] \lambda_u^2$$

$$= \frac{\ln(b/a)}{4\pi\epsilon_0} \frac{Q^2}{\left(L - h + \frac{\epsilon}{\epsilon_0} h\right]}$$

take L large and define $\lambda_0 = Q/L$

$$W_{TOT} \cong \frac{\ln(b/a)}{4\pi\epsilon_0} \frac{Q^2}{L} \left[1 - \frac{h}{L}\left(\frac{\epsilon}{\epsilon_0} - 1\right) \right]$$

$$F_{em} = \frac{\partial W_{TOT}}{\partial h} = -\frac{\ln(b/a)}{4\pi\epsilon_0} \lambda_0^2 \left(\frac{\epsilon}{\epsilon_0} - 1\right)$$

$\Rightarrow$ energy decreases for $\epsilon > \epsilon_0$

$\Rightarrow$ upward force

$$F_g = \pi(b^2 - a^2)\rho g h \qquad \epsilon = \epsilon_0(1 + \chi_e)$$

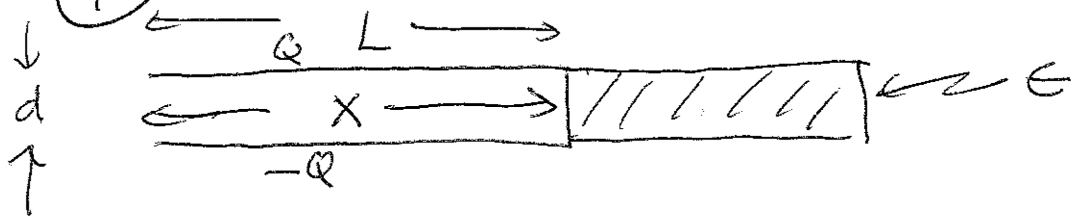
$$\Rightarrow \quad \pi(b^2 - a^2)\rho g h = \frac{\ln(b/a)}{4\pi\epsilon_0} \lambda_0^2 \chi_e$$

$$V = \int_a^b dr\, E = \frac{\lambda_0}{2\pi\epsilon_0} \ln\frac{b}{a}$$

$$\Rightarrow \quad \chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}$$

a) The dielectric is pulled into the capacitor and oscillates periodically

4



b) Find stored energy for $0<x<L$.
Charge moves so E is a constant along the capacitor $\Rightarrow$ σ/ϵ is constant

$$\Rightarrow \sigma_{\epsilon_0},\ \sigma_{\epsilon}$$

$$\sigma_{\epsilon_0} x L + \sigma_{\epsilon} L(L-x) = Q = \sigma_0 L^2$$

$$\frac{\sigma_{\epsilon_0}}{\epsilon_0} = \frac{\sigma_{\epsilon}}{\epsilon}$$

$$\sigma_{\epsilon_0}\left[xL + \frac{\epsilon}{\epsilon_0}(L-x)L\right] = \sigma_0 L^2 = Q$$

$$W_{TOT} = \frac{1}{2}\int d\underset{\sim}{x}\, \epsilon E^2 \qquad E = \frac{\sigma_{\epsilon_0}}{\epsilon_0}$$

$$= \frac{1}{2} E^2 \left[\epsilon(L-x) + \epsilon_0 x\right] L d$$

$$= \frac{1}{2} Q^2 \frac{Ld}{\left[\epsilon(L-x)+\epsilon_0 x\right] L^2}$$

$$W_{TOT} = \frac{1}{2} Q^2 \frac{d}{L} \frac{1}{\epsilon(L-x)+\epsilon_0 x}$$

$$F = -\frac{\partial W_{TOT}}{\partial x} = -\frac{1}{2} Q^2 \frac{d}{L} \frac{(\epsilon-\epsilon_0)}{\left[\epsilon(L-x)+\epsilon_0 x\right]^2}$$

$$m\ddot{x} = -\frac{1}{2} Q^2 \frac{d}{L} \frac{\epsilon-\epsilon_0}{\left[\epsilon(L-x)+\epsilon_0 x\right]^2}$$

c) Use energy conservation

ΔW = change in electric field energy

$$= W_{TOT}(x=0) - W_{TOT}(x=L)$$

$$= \frac{1}{2} Q^2 \frac{d}{L} \left[\frac{1}{\epsilon L} - \frac{1}{\epsilon_0 L} \right]$$

$$= \frac{1}{2} Q^2 \frac{d}{L^2} \frac{\epsilon_0 - \epsilon}{\epsilon_0 \epsilon}$$

$$\frac{1}{2} m v^2 = -\Delta W$$

$$v = \sqrt{\frac{Q^2}{m} \frac{d}{L^2} \frac{\epsilon - \epsilon_0}{\epsilon_0 \epsilon}}$$

d) ~~Let~~

$$\frac{1}{2} m v^2 = W_{TOT}(x=L) - W_{TOT}(x)$$

$$= \frac{1}{2} Q^2 \frac{d}{L} \left[\frac{1}{\epsilon_0 L} - \frac{1}{\epsilon(L-x) + \epsilon_0 x} \right]$$

$$\tau = 4 \int_L^0 \frac{d\tau}{dx} dx = 4 \int_L^0 \frac{dx}{v}$$

For $\epsilon = \epsilon_0(1+\delta)$

For $\delta \ll 1$, easier to go back to equation of motion

$$m\ddot{x} \simeq -\frac{1}{2} Q^2 \frac{d}{L} \frac{\epsilon_0 \delta}{\epsilon_0^2 L^2}$$

$$\ddot{x} = -\frac{1}{2} \frac{Q^2}{m} \frac{d}{\epsilon_0 L^3} \delta$$

$$= \text{const.}$$

Let $a_x = \frac{1}{2} \frac{Q^2}{m} \frac{d}{\epsilon_0 L^3} \delta$

To go a distance $x = L$

$$L = \frac{1}{2} a_x t^2 \Rightarrow \tau = 4t$$

$$\tau = 4 \sqrt{\frac{2L}{a_x}}$$

$$= 4 \sqrt{\frac{2 L \epsilon_0 L^3 m}{\frac{1}{2} Q^2 d \delta}}$$

$$\boxed{\tau = \frac{8L^2}{Q} \sqrt{\frac{\epsilon_0 m}{d \delta}}}$$

For $\delta \gg 1$, ω_{TOT} quickly drops to zero so the dielectric has a constant velocity

$$V \simeq \frac{Q}{L} \sqrt{\frac{d}{m \epsilon_0}}$$

$$\Rightarrow Vt = L \qquad \tau = 4t$$

$$\boxed{\tau = 4 \frac{L^2}{Q} \sqrt{\frac{m \epsilon_0}{d}}}$$