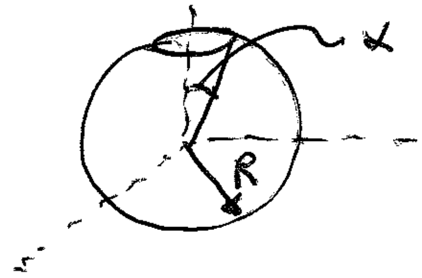
Physics 606 Homework #4 Spring '12

Dr. Drake

1. Jackson 3.2
2. Jackson 3.6
3. Jackson 3.20
4. Consider an infinite cylindrical conductor of radius "a" which is sliced at $z = \pm L/2$. The segment $|z| < L/2$ is maintained at a potential V. The remainder of the cylinder is grounded.
   (a) Sketch the electric field E in the region $\rho = \sqrt{x^2 + y^2} > a$.
   (b) Considern the limit in which $a >> L$. Sketch the electric field in the region $\rho > a$ in this limit. Estimate the magnitude of the electric field at the surface in the region $|z| \sim 0$ and the force per unit area acting on the conductor in this region. What is the direction of the force? Over what scale length does the potential fall off in the radial and axial directions?
   (c) Derive an exact expression for the potential Φ in the region $\rho > a$ for arbitrary L/a.
   (d) Now take the limiting case where $a \gg L$. What is the characteristic scale length remaining in the problem? What does the solution represent? Explicitly evaluate the radial electric field just outside of the cylinder for $z \sim 0$ to check your previous estimate.

Physics 606 Hwk 4

(3.2) Use the infinite medium Green's function in spherical coordinates



$\sigma = \frac{Q}{4\pi R^2}$ except for $\theta < \alpha$

charge at radius R

$$\Phi = \frac{1}{4\pi\epsilon_0} \int d\underset{\sim}{x}' \, \rho(\underset{\sim}{x}') \, G(\underset{\sim}{x}, \underset{\sim}{x}')$$

$$\rho = \frac{Q}{4\pi R^2} \, \delta(r-R) \, H(\cos\alpha - \cos\theta)$$

$$G = 4\pi \sum_l \sum_m \frac{1}{2l+1} Y^*_{lm}(\theta', \varphi') \, Y_{lm}(\theta, \varphi) \, \frac{r^l}{R^{l+1}}$$

for $r < R$ $\qquad \int d\underset{\sim}{x}' = \int_0^\infty dr' \, d\cos\theta' \, r'^2 \, d\varphi'$

$\Rightarrow$ integral over φ' eliminates all but $m = 0$

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{4\pi \cancel{R^2}} \cancel{\int} \cancel{R^2} \, 2\pi \sum_l \cancel{\frac{1}{2l+1}} \frac{\cancel{2l+1}}{4\pi}$$

$$\times \int_{-1}^{\cos\alpha} d\cos\theta' \, P_l(\cos\theta') \, P_l(\cos\theta) \, \frac{r^l}{R^{l+1}}$$

$$= \frac{Q}{8\pi\epsilon_0} \sum_l \frac{r^l}{R^{l+1}} P_l(\cos\theta) \int_{-1}^{\cos\alpha} d\cos\theta' \, P_l(\cos\theta')$$

$$\int_{-1}^{\cos\alpha} d(\cos\theta')\, P_\ell(\cos\theta') = \frac{1}{2\ell+1}\left[P_{\ell+1}\Big|_{-1}^{\cos\alpha} - P_{\ell-1}\Big|_{-1}^{\cos\alpha} \right]$$

$$P_{\ell+1}(-1) = P_{\ell-1}(-1)$$

$$\Rightarrow \frac{1}{2\ell+1}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]$$

$$\Rightarrow \cos\alpha + 1 \quad \text{for } \ell = 0$$

$$\Rightarrow P_{-1}(\cos\alpha) = -1$$

$$\boxed{\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{r^\ell}{(2\ell+1)R^{\ell+1}} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right] P_\ell(\cos\theta)}$$

outside is the same except for

$$r^\ell \to R^\ell$$

$$R^{\ell+1} \to r^{\ell+1}$$

(b) By symmetry at the origin $\underline{E}$ will be in the z direction

$$E_z = -\frac{\partial \Phi}{\partial z}\Big|_{\substack{r=0 \\ \cos\theta = 0}} \quad \Rightarrow \text{only } \ell = 1 \text{ survives}$$

$$\Rightarrow z = r\cos\theta$$

$$E_z = -\frac{Q}{8\pi\epsilon_0}\frac{1}{3}\frac{1}{R^2}\left[\underbrace{P_2(\cos\alpha) - P_0(\cos\alpha)}_{\frac{3\cos^2\alpha - 1}{2} - 1}\right]$$

$$\boxed{E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{R^2}\left(1 - \cos^2\alpha\right)}$$

(c)

<u>small cap</u> $\Rightarrow \cos\alpha \to 1$

$$P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) = 0$$

except for $\ell = 0$

$$\Phi = \frac{Q}{8\pi\epsilon_0}\frac{1}{R}\,2 = \frac{Q}{4\pi\epsilon_0 R}$$

$\Rightarrow$ same as uniform surface charge

$$E_z \Rightarrow 0$$

<u>large cap</u>

$\Phi \to 0$ since total charge $\to 0$

$$E_z = \frac{Q}{8\pi\epsilon_0}\frac{1}{R^2}(1-\cos\alpha)(1+\cos\alpha)$$

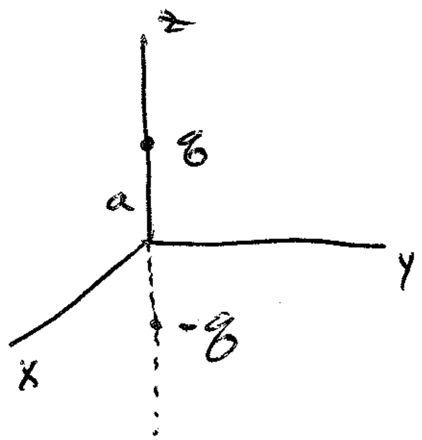
$$\simeq \frac{Q_{tot}}{4\pi\epsilon_0}\frac{1}{R^2}$$

$$Q_{tot} = \frac{1+\cos\alpha}{2}\,Q$$

$=$ total charge.

$\Rightarrow$ E_z is from Q_{tot} at a distance R.

3.6



needed since only $\frac{1}{2}$ of the δ-function is integrated for $\cos\theta \in (-1,1)$

$$\rho = \frac{2q\,\delta(r-a)\left[\delta(\cos\theta-1) - \delta(\cos\theta+1)\right]}{2\pi a^2}$$

$$\frac{\partial}{\partial\varphi} = 0$$

$\Rightarrow$ expand potential directly

$$\Phi = \sum_\ell c_\ell \, P_\ell(\cos\theta)\, R_\ell(r)$$

$$\nabla^2\Phi = \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right]\Phi = -\frac{\rho}{\epsilon_0}$$

$$\sum_\ell c_\ell \, P_\ell(\cos\theta)\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \ell(\ell+1)\right] R_\ell = -\frac{\rho}{\epsilon_0}$$

operate with $\int_{-1}^{1} d\cos\theta \, P_{\ell'}(\cos\theta)$ to eliminate $\sum_\ell$

$$c_\ell \frac{2}{2\ell+1}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \ell(\ell+1)\right] R_\ell = -\frac{2q\,\delta(r-a)}{2\pi a^2 \epsilon_0}$$

$$\otimes \frac{1}{2}\Big[\underbrace{P_\ell(1) - P_\ell(-1)}\Big]$$

2 for ℓ odd
0 for ℓ even

$R \sim r^{\ell}, r^{-\ell-1}$

$$R_\ell = \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$r_<$ = smaller of r, a

$r_>$ = larger of r, a

$\Rightarrow$ jump condition

$$c_\ell \left. \frac{\partial R_\ell}{\partial r} \right|_{a-\epsilon}^{a+\epsilon} = -\frac{(2\ell+1)}{2\pi a^2 \epsilon_0} q$$

$$c_\ell \left[+(\ell+1) \frac{1}{\cancel{a^2}} + \ell \frac{1}{\cancel{a^2}} \right] = + \frac{(2\ell+1)}{2\pi \cancel{a^2} \epsilon_0} q$$

$$c_\ell = \frac{1}{2\pi\epsilon_0} q$$

$$\boxed{\Phi = \frac{q}{2\pi\epsilon_0} \sum_{\ell\ \text{odd}} P_\ell(\cos\theta) \frac{r_<^{\ell}}{r_>^{\ell+1}}}$$

b) Take $a \to 0$ with $qa \equiv \frac{p}{2}$

$$\Phi = \frac{q}{2\pi\epsilon_0} \sum_{\ell\ \text{odd}} P_\ell(\cos\theta) \frac{a^{\ell}}{r^{\ell+1}}$$

$$\boxed{\Phi = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}}$$

c) Note that a uniform electric field along z, $\underset{\sim}{E} = E_0 \hat{k}$, has a potential

$$\Phi_0 = -E_0 z = -E_0 r\cos\theta$$

Total potential at a radius, b, is

$$\Phi_{tot} = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{b^2} - E_0 b\cos\theta$$

$\Rightarrow$ want this constant

$$\Rightarrow \quad E_0 = \frac{p\cos\theta}{4\pi\epsilon_0 b^3}$$

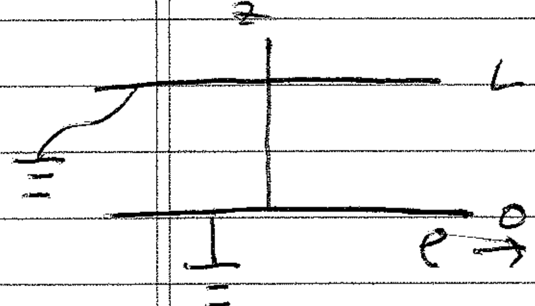
For $r < b$,

$$\Phi = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} - \frac{p\cos\theta}{4\pi\epsilon_0 b^3} r\cos\theta$$

$$\boxed{\Phi = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\left(1 - \frac{r^3}{b^3}\right)}$$

3.20 Take the charge to be at radius ϵ and then let $\epsilon \to 0$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\, q\, \frac{\delta(\rho-\epsilon)}{2\pi\rho}\,\delta(z-z_0)$$



Choose basis functions $\sin\left(\frac{n\pi z}{L}\right)$ in z.

$$\Phi = \sum_n R_n(\rho)\,\sin\left(\frac{n\pi z}{L}\right)$$

$$\sum_n \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_n - \frac{n^2\pi^2}{L^2}R_n\right]\sin\frac{n\pi z}{L} = -\frac{1}{\epsilon_0}\, q\, \frac{\delta(\rho-\epsilon)}{2\pi\rho}$$

$$\otimes\ \delta(z-z_0)$$

mult by $\sin\left(\frac{n'\pi z}{L}\right)$ and integrate $(0, L)$

$\Rightarrow$ eliminates sum over n

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_n - \frac{n^2\pi^2}{L^2}R_n\right)\frac{1}{2}L = -\frac{q}{\epsilon_0}\,\frac{\delta(\rho-\epsilon)}{2\pi\rho}\sin\frac{n\pi z_0}{L}$$

$\Rightarrow \nu = 0$ Bessel's Eqn

jump conditions

$$\epsilon\,\cancel{\rho}\,\frac{\partial}{\partial\rho}R_n\Big|_{\epsilon_-}^{\epsilon_+} = -\frac{2q}{L\epsilon_0}\,\cancel{q}\,\frac{1}{2\pi\cancel{\epsilon}}\sin\left(\frac{n\pi z_0}{L}\right)$$

$$R_n\Big|_{\epsilon_-}^{\epsilon_+} = 0$$

$\rho > \epsilon_0$:

bounded solution as $\rho \to \infty$

$$R_n = c_n I_0(k_n \epsilon) K_0(k_n \rho)$$

$\rho \le \epsilon_0$

bounded solution as $\rho \to 0$

$$R_n = c_n I_0(k_n \rho) K_0(k_n \epsilon)$$

$$k_n \epsilon c_n \left[I_0(k_n \epsilon) K_0'(k_n \epsilon) - I_0'(k_n \epsilon) K_0(k_n \epsilon) \right]$$

$$= -\frac{2q}{L\epsilon_0} \frac{1}{2\pi} \sin\left(\frac{n\pi z_0}{2}\right)$$

$$k_n \epsilon c_n \left[\frac{1}{\Gamma(1)} \left(-\frac{1}{k_n \epsilon} \right) - 0 \right]$$

$$c_n = \frac{\cancel{2} q}{L\epsilon_0} \frac{1}{\cancel{2}\pi} \sin\left(\frac{n\pi z_0}{L}\right)$$

$$\Phi = \frac{q}{\pi\epsilon_0 L} \sum_n \sin\left(\frac{n\pi z_0}{L}\right) \sin\left(\frac{n\pi z}{L}\right) K_0\left(\frac{n\pi \rho}{L}\right)$$

(b) <u>lower plane</u> $\quad E_z = -\frac{\partial \Phi}{\partial z}\Big|_{z=0} \qquad \sigma_0 = \epsilon_0 E_z$

$$E_z = -\frac{q}{\cancel{\pi}\epsilon_0 L} \sum_n \frac{n\cancel{\pi}}{L} \sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

$$\sigma_0 = -\frac{q}{\cancel{\pi} L^2} \sum_n \sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

<u>upper plane</u> $\quad \sigma_L = -\epsilon_0 E_z$

$$E_z = -\frac{q}{\cancel{\pi}\epsilon_0 L} \sum_n \frac{n\cancel{\pi}}{L} \sin\left(\frac{n\pi z_0}{L}\right) (-1)^n K_0\left(\frac{n\pi\rho}{L}\right)$$

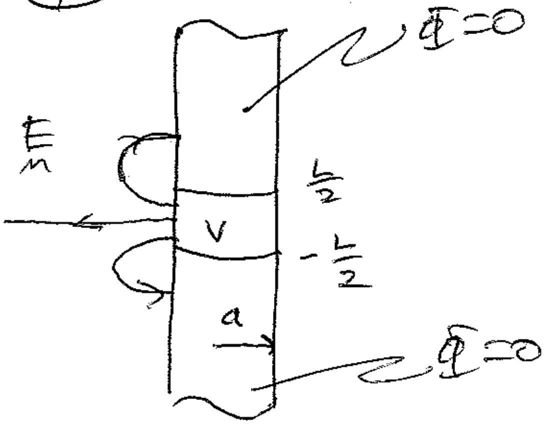
$$\sigma_L = \frac{q}{L^2} \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

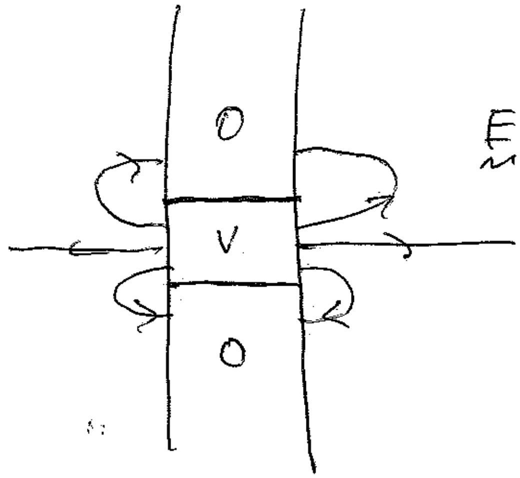
(c)
$$Q_L = \int_0^\infty d\rho \, 2\pi\rho \, \sigma_L$$

$$= \frac{2\cancel{\pi}}{\cancel{L^2}} q \sum_n \frac{\cancel{L^2}}{n^2\pi^2} \sin\left(\frac{n\pi z_0}{L}\right) \int_0^\infty ds \, s \, K_0(s)$$

$$= \frac{2q}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi z_0}{L}\right)$$

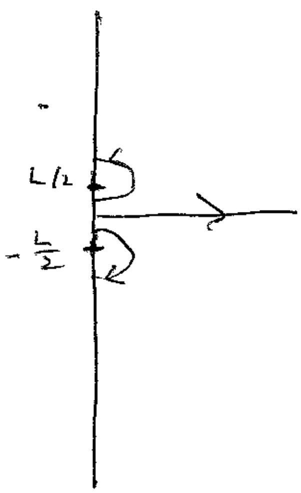
4 a)





b) $a \gg L$

$\Rightarrow$ like infinite plane with a strip of potential V



$\Rightarrow$ scale length above surface where $\Phi \neq 0$ is $\sim L$

$\Rightarrow$ $E \sim \frac{V}{L}$

pressure $\sim \frac{1}{2} \epsilon_0 E^2$

$\sim \frac{1}{2} \epsilon_0 \frac{V^2}{L^2}$

$\Rightarrow$ <u>outward force</u>

$\Rightarrow$ scale length L in both z and $r-a$.

c) potential in region $\rho > a$

$\Rightarrow$ solve Laplace's Eqn with $\frac{\partial}{\partial \varphi} = 0$

$$\Phi = \int_0^\infty dk \cos kz \, K_0(k\rho) \, g_k$$

$\Rightarrow$ even around $z = 0$

$\Rightarrow$ $K_0 \to 0$ as $\rho \to \infty$

$\Rightarrow$ match BC at $\rho = a$

$$\Phi(\rho = a, z) = \int_0^\infty dk \cos kz \, K_0(ka) \, g_k$$

multiply by $\cos k'z$ and integrate over z

$\Rightarrow$ note k, k' both positive

$$V \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \cos k'(z) = V \left. \frac{\sin k'z}{k'} \right|_{-\frac{L}{2}}^{\frac{L}{2}} = 2V \frac{\sin k'\frac{L}{2}}{k'}$$

$$= \int_0^\infty dk \, g_k \, \cancel{\cos kz} \, K_0(ka) \underbrace{\int_{-\infty}^{\infty} dz \cos k'z \cos kz}_{I}$$

$$I = \int_{-\infty}^{\infty} dz \left(\frac{e^{ik'z} + e^{-ik'z}}{2} \right) \left(\frac{e^{ikz} + e^{-ikz}}{2} \right)$$

$$= \frac{\pi}{2} \left[2 \, \overset{0}{\cancel{\delta(k + k')}} + 2 \, \delta(k - k') \right] = \pi \, \delta(k - k')$$

$$2V \frac{\sin\frac{kL}{2}}{k} = \pi g_k K_0(ka)$$

$$g_k = \frac{2V}{\pi} \frac{\sin(kL/2)}{k K_0(ka)}$$

$$\boxed{\Phi = \frac{2V}{\pi} \int_0^\infty dk \frac{\sin(kL/2)}{k} \cos kz \frac{K_0(k\rho)}{K_0(ka)}}$$

d) Take $a \gg L$ and $\rho > a$

$k\rho \sim ka \gg 1 \Rightarrow$ expand K_0 for large argument

$$\Phi = \frac{2V}{\pi} \int_0^\infty dk \frac{\sin(\frac{kL}{2})}{k} \cos kz \, e^{-k(\rho - a)}$$

remaining scale length is L

Let $\frac{kL}{2} \equiv s$

$$\Phi = \frac{2V}{\pi} \int_0^\infty ds \frac{\sin s}{s} \cos\left(\frac{2z}{L} s\right) e^{-\frac{(\rho-a)}{L} 2s}$$

For $z = 0$ ~~...~~

$$\Phi = \frac{2V}{\pi} \int_0^\infty ds \frac{\sin s}{s} e^{-(\rho - a)\frac{2s}{L}}$$

$$e' \equiv e - a$$

$$E_e = -\frac{\partial \Phi}{\partial e} = \frac{4V}{\pi L}\int_0^\infty ds\ \sin(s)\ e^{-e'\frac{2s}{L}}$$

$$= \frac{4V}{\pi L}\,\mathrm{Im}\int_0^\infty ds\ e^{is}\ e^{-e'\frac{2}{L}s}$$

$$= \frac{4V}{\pi L}\,\mathrm{Im}\ \left.\frac{e^{is}e^{-e'\frac{2}{L}s}}{i - \frac{e'2}{L}}\right|_0^\infty$$

$$= -\frac{4V}{\pi L}\,\mathrm{Im}\left(\frac{1}{i - \frac{2}{L}e'}\right)$$

$$= -\frac{4V}{\pi L}\,\mathrm{Im}\ \frac{-i - \frac{2}{L}e'}{1 + \frac{4}{L^2}e'^2}$$

$$\boxed{E_e = \frac{4V}{\pi L}\ \frac{1}{1 + \frac{4}{L^2}e'^2}}$$

$$= \frac{4V}{\pi L} \quad \text{for } e' = 0$$

$$\sim \frac{V}{L}$$