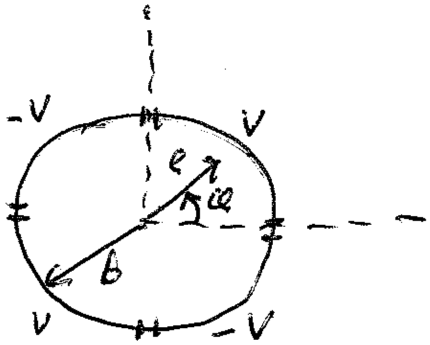
2.14



long conducting cylinder

For $\rho < b$, satisfy

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\Phi = 0$$

$$\Phi = \Sigma Q(\varphi)\, R(\rho)$$

$\Rightarrow$ choose oscillatory in φ so satisfy periodicity requirement.

$$Q(\varphi) \sim e^{\pm i m \varphi}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R - \frac{m^2}{\rho^2}R = 0$$

$$\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R - m^2 R = 0 \quad \Rightarrow R \sim \rho^{\pm m}$$

~~R is finite at ρ=0 ⇒ R ~ ρ^m~~ $\sim \rho^m$ $\Rightarrow$ regular at $\rho = 0$

Q is odd around $\varphi = 0$

$$\Rightarrow Q \sim \sin m\varphi$$

$$\Phi\left(\varphi + \tfrac{\pi}{2}\right) = -\Phi(\varphi)$$

$$\Rightarrow \sin m\left(\varphi + \tfrac{\pi}{2}\right) = -\sin m\varphi$$

$$\sin(m\varphi)\cos\left(\tfrac{m\pi}{2}\right) + \cos m\varphi \sin\tfrac{m\pi}{2} = -\sin m\varphi$$

$\cos\left(\frac{m\pi}{2}\right) = -1 \quad \Rightarrow \quad \frac{m\pi}{2} = (2n+1)\pi$

$$m = 2(2n+1) \quad \text{with } n = 0, 1, 2 \ldots$$

$$\Phi = \sum_{n=0}^{\infty} c_n \left(\frac{\rho}{b}\right)^{4n+2} \sin(4n+2)\varphi$$

BC at $\rho = b$

$$\Phi(\rho = b, \varphi) = \sum_{n=0}^{\infty} c_n \sin(4n+2)\varphi$$

multiply by $\sin(4n'+2)\varphi$ and integrate

$$\int_0^{2\pi} d\varphi \, \sin(4n'+2)\varphi \, \Phi = c_{n'} \frac{1}{2} 2\pi$$

$$V 4 \int_0^{\frac{\pi}{2}} d\varphi \, \sin(4n'+2)\varphi = c_{n'} \pi$$

$$-4V \left. \frac{\cos(4n'+2)\varphi}{4n'+2} \right|_0^{\pi/2} = c_{n'} \pi$$

$$\cos(4n'+2)\frac{\pi}{2} = \cos\pi = -1$$

$$\frac{1}{\pi} \frac{8V}{4n'+2} = c_{n'}$$

$$\boxed{\Phi = \sum_n \frac{8V}{\pi \, (4n+2)} \left(\frac{\rho}{b}\right)^{4n+2} \sin(4n+2)\varphi}$$

b) Let $S' = \sum_n \frac{1}{\cancel{4n+2}\; 2n+1} \left(\frac{\rho}{b}\right)^{4n+2} \sin(4n+2)\varphi$

$$= \mathrm{Im} \sum_n \frac{1}{2n+1} z^{2n+1}$$

$$z = \left(\frac{\rho}{b}\right)^2 e^{i2\varphi}$$

$$\frac{dS}{dz} = \mathrm{Im} \sum_n z^{2n} = \mathrm{Im}\left(1 + z^2 + z^4 + \dots\right)$$

$$= \mathrm{Im} \frac{1}{1-z^2}$$

$$s' = \text{Im}\,\frac{1}{2}\int dz\left(\frac{1}{1-z}+\frac{1}{1+z}\right)$$

$$= \frac{1}{2}\,\text{Im}\,\ln\left(\frac{1+z}{1-z}\right)$$

$$\frac{1+z}{1-z} = \frac{(1+z)(1-z^*)}{|1-z|^2} = \frac{1+2\,\text{Im}\,z-|z|^2}{|1-z|^2}$$

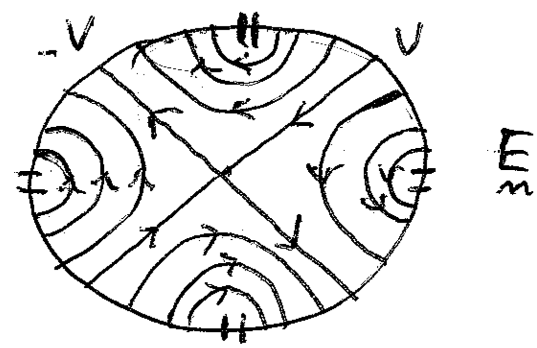
$$= \left|\frac{1+z}{1-z}\right| e^{i\Theta}$$

$$\Theta = \tan^{-1}\frac{2\,\text{Im}\,z}{1-|z|^2}$$

$$= \tan^{-1}\frac{2\left(\frac{\rho}{b}\right)^2\sin 2\varphi}{1-\frac{\rho^4}{b^4}}$$

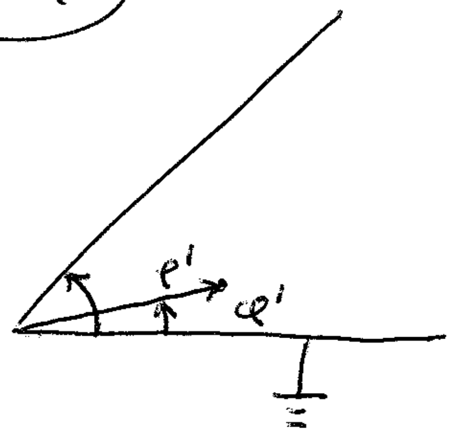
$$\boxed{\Phi = \frac{2V}{\pi}\tan^{-1}\left(\frac{2b^2\rho^2\sin 2\varphi}{b^4-\rho^4}\right)}$$

c)



Potential contours are $\perp$ to lines of $\underset{\sim}{E}$.

2.25a



$$\nabla^2 G = -4\pi \frac{\delta(\varphi-\varphi')\,\delta(\rho-\rho')}{\rho'}$$

Note: $\int d\underset{\sim}{x}' \, \frac{\delta(\varphi-\varphi')\,\delta(\rho-\rho')}{\rho'} = 1$

Let $G = \sum_n g_n(\rho) \sin(\alpha_n \varphi)$

as in 2.14 where $G(\varphi=0) = 0$

and $\sin(\alpha_n \beta) = 0$

$$\Rightarrow \alpha_n \beta = n\pi \Rightarrow \alpha_n = \frac{n\pi}{\beta}$$

$$\sum_n \sin \alpha_n \varphi \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} g_n - \frac{1}{\rho^2} \alpha_n^2 g_n \right) = -4\pi \frac{\delta(\varphi-\varphi')\,\delta(\rho-\rho')}{\rho'}$$

multiply by $\sin \alpha_{n'} \varphi$ and integrate over φ

$$\frac{1}{2}\beta \left(\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} g_n - \frac{1}{\rho} \alpha_n^2 g_n \right) = -4\pi \sin(\alpha_n \varphi')\,\delta(\rho-\rho')$$

<u>$\rho \neq \rho'$</u> $\quad g_n \sim \rho^{\pm \alpha_n}$

$$\left.\begin{array}{ll} \rho < \rho' & g_n = c_n \rho'^{-\alpha_n} \rho^{\alpha_n} \\ \rho > \rho' & g_n = c_n \rho^{-\alpha_n} \rho'^{\alpha_n} \end{array}\right\} \quad g_n \Big|_{\rho'-\epsilon}^{\rho'+\epsilon} = 0$$

near $\rho = \rho'$

$$\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} g_n = -\frac{8\pi}{\beta} \sin(\alpha_n \ell') \, \delta(\rho - \rho')$$

$$\left. \frac{\partial g_n}{\partial \rho} \right|_{\rho' - \epsilon}^{\rho' + \epsilon} = -\frac{8\pi}{\beta \rho'} \sin(\alpha_n \ell')$$

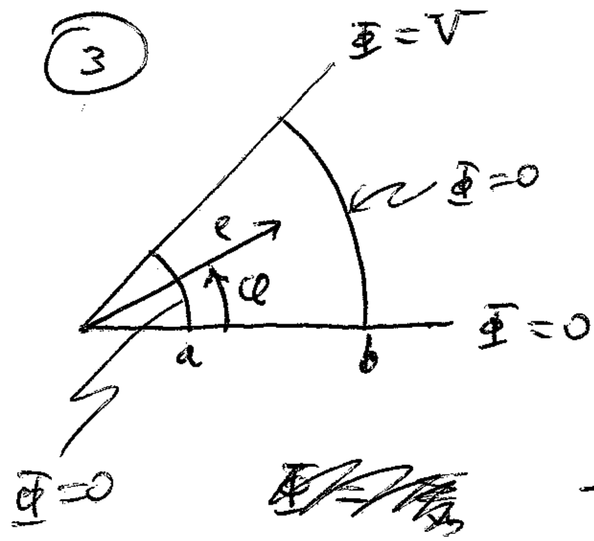
$$C_n \alpha_n \left[-\frac{1}{\rho'} - \frac{1}{\rho'} \right] = -\frac{8\pi}{\beta} \frac{1}{\rho'} \sin(\alpha_n \ell')$$

$$C_n = \frac{4\pi}{\beta \alpha_n} \sin(\alpha_n \ell')$$

$$\Phi = \frac{4\pi}{\beta} \sum_n \frac{1}{\alpha_n} \rho_<^{\alpha_n} \rho_>^{-\alpha_n} \sin(\alpha_n \ell) \sin(\alpha_n \ell')$$

$$\alpha_n = \frac{n\pi}{\beta}$$

3



$$\Phi = \sum_n R_n(\rho)\,\sinh(\alpha_n \varphi)$$

$\Rightarrow$ exponential in φ with $\Phi = 0$ at $\varphi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} R_n + \frac{1}{\rho^2}\alpha_n^2 R_n = 0$$

define variable $s = \ln\left(\frac{\rho}{a}\right)$

$$\frac{d}{ds} = \frac{d\rho}{ds}\frac{d}{d\rho} \qquad \frac{ds}{d\rho} = \frac{1}{\rho}$$

$$= \rho \frac{d}{d\rho}$$

$$\frac{d^2}{ds^2} R_n(s) + \alpha_n^2 R_n = 0$$

$$R_n(s) \sim \sin \alpha_n s,\ \cos \alpha_n s$$

Want $R_n(\rho = a) = R_n(s = 0) = 0$

$$\Rightarrow R_n(s) \sim \sin \alpha_n s$$

Want $R_n(\rho = b) = R_n[\ln(b/a)] = 0$

$$\alpha_n \ln\left(\frac{b}{a}\right) = n\pi$$

$$\boxed{\alpha_n = \frac{n\pi}{\ln(b/a)}}$$

$$\Phi = \sum_n c_n \sin(\alpha_n s) \sinh(\alpha_n \varphi)$$

$$\Phi(\varphi = \beta) = \sum_n c_n \overset{\sin}{\wedge}(\alpha_n s) \sinh(\alpha_n \beta)$$

$$\parallel$$

$$V$$

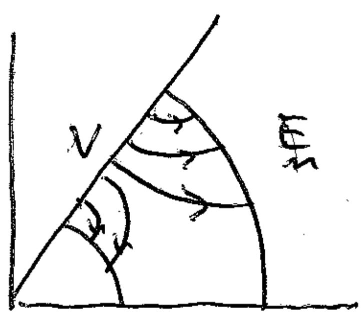
$$V \int_0^{\ln(b/a)} ds \sin(\alpha_n s) = c_n \frac{1}{2} \sinh(\alpha_n \beta) \ln(b/a)$$

$$\parallel$$

$$V - \frac{\cos \alpha_n s}{\alpha_n} \Big|_0^{\ln(b/a)}$$

$$\parallel$$

$$+V\left[\frac{-\cos \alpha_n \ln(b/a) + 1}{\alpha_n}\right] = \cancel{} + \frac{V}{\alpha_n}\left(-\cos(n\pi) + 1\right)$$



$$= +\frac{2V}{\alpha_n} \quad n \text{ ~~even~~ odd}$$

$$= 0 \quad n \text{ ~~odd~~ even}$$

$$\Phi = \frac{4V}{\ln(b/a)} \sum_{n \text{ odd}} \frac{1}{\alpha_n} \sin \alpha_n \ln\left(\frac{\rho}{a}\right) \frac{\sinh(\alpha_n \varphi)}{\sinh(\alpha_n \beta)}$$

$$\alpha_n = \frac{n\pi}{\ln(b/a)}$$

Onthogonality

$$\int_0^{\ln(b/a)} ds \sin(\alpha_n s) \sin \alpha_{n'} s = \frac{1}{2} \ln(b/a) \delta_{nn'}$$

Satisfies S-L equation with 0 BCs.

4

Grounded conducting box with dimensions a, a, c. For charge q ~~in the~~ at $a/2, a/2$ and $x = a/2$, $y = a/2$

$$\Phi = \sum_{\substack{m,n \\ \text{odd}}} \frac{4q}{a^2 \gamma_{mn} \epsilon_0} \cancel{\sin^2\left(\frac{n\pi}{2}\right)} \cancel{\sin^2\left(\frac{m\pi}{2}\right)}$$

$$\otimes \quad \frac{1}{\sinh(\gamma_{mn} c)} \sinh(\gamma_{mn} z_<) \sinh \gamma_{mn}(c - z_>)$$

$$z_< = \text{smaller of } z, z_0$$
$$z_> = \text{larger of } z, z_0$$
$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}(n^2 + m^2)$$

$\Rightarrow$ only m, n odd contribute.

a) By symmetry the charges on the boundaries produce forces which cancel.

b) Calculate the electric field acting on the charge for a small displacement $\Delta z \ll c$ from $c/2$.

$\Rightarrow$ Eliminate self field by averaging the E_z fields for $z = z_0 + \epsilon$ and $z = z_0 - \epsilon$

$$E_z = -\frac{4q}{a^2 \epsilon_0} \sum_{\substack{m,n \\ \text{odd}}} \frac{1}{\cancel{\gamma_{mn}}} \frac{\cancel{\gamma_{mn}} \ \frac{1}{2}}{\sinh(\gamma_{mn} c)} \Big[-\cosh \gamma_{mn}(c - z_0) \sinh \gamma_{mn} z_0$$
$$+ \cosh \gamma_{mn} z_0 \sinh \gamma_{mn}(c - z_0) \Big]$$

$$= -\frac{4q}{a^2 \epsilon_0} \sum_{\substack{m,n \\ \text{odd}}} \frac{1}{2} \frac{\sinh \gamma_{mn}(c - 2z_0)}{\sinh(\gamma_{mn} c)} \qquad z_0 = \frac{c}{2} + \Delta z$$

$$= +\frac{2q}{a^2 \epsilon_0} \sum_{\substack{m,n \\ \text{odd}}} \frac{\sinh \gamma_{mn} \Delta z}{\sinh(\gamma_{mn} c)}$$

For small Δz

$$E_z \simeq \frac{2q}{a^2\epsilon_0} \sum_{\substack{m,n \\ \text{odd}}} \gamma_{mn} \frac{\Delta z}{\sinh(\gamma_{mn} c)}$$

<u>$a/c \ll 1$</u> : $\Rightarrow$ $\gamma_{mn} c \gg 1$ for all m, n

$$E_z \simeq \frac{2q}{a^2\epsilon_0} \sum_{\substack{m,n \\ \text{odd}}} \gamma_{mn} \frac{\Delta z \; 2}{e^{\gamma_{mn} c}}$$

$$\simeq \frac{4q}{a^2\epsilon_0} \gamma_{11} e^{-\gamma_{11} c} \Delta z \qquad \text{with } \gamma_{11} = \frac{\pi}{a}\sqrt{2}$$

$$E_z = \frac{4q}{a^2\epsilon_0} \frac{\pi}{a}\sqrt{2}\, e^{-\frac{\pi}{a}\sqrt{2} c} \Delta z$$

$\Longrightarrow$ note E_z is small because most of the image charge is on the side walls, which don't produce much E_z.

$\Rightarrow$ no E_z as $c \to \infty$.

<u>$a/c \gg 1$</u> : For this case m,n must be large before $\sinh(\gamma_{mn} c)$ cuts off the sum.

$\Rightarrow$ change from discrete sum to a continuum.

Write $m = 2m' + 1$ with $m' = 0, 1, 2 \ldots$

$n = 2n' + 1$ with $n' = 0, 1, 2 \ldots$

Let $k_m \equiv \frac{\pi}{a}(2m'+1) \Rightarrow dk_m = \frac{2\pi}{a} dm'$

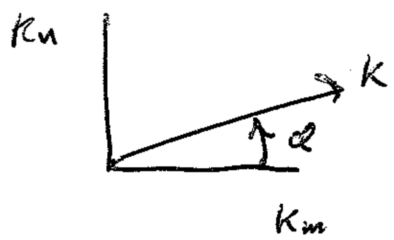
$k_n = \frac{\pi}{a}(2n'+1) \qquad dk_n = \frac{2\pi}{a} dn'$

$$\sum_{m'} \Rightarrow \sum_{m'} dm' = \frac{a}{2\pi} \int_0^\infty dk_m$$

$$E_z = \frac{2q}{\cancel{a^2}\epsilon_0}\Delta z \int_0^\infty dk_m dk_n \; k \frac{1}{\sinh kc} \frac{\cancel{a^2}}{4\pi^2}$$

$$k^2 = k_m^2 + k_n^2$$

$$dk_m dk_n = k\, dk\, d\varphi$$



$$E_z = \frac{q\,\Delta z}{2\pi^2\epsilon_0}\int_0^{\pi/2} d\varphi \int_0^\infty dk\, k^2 \frac{1}{\sinh kc}$$

$$= \frac{q\,\Delta z}{2\pi^2\epsilon_0}\,\frac{\pi}{2}\,\frac{1}{c^3}\int_0^\infty ds\; s^2 \frac{1}{\sinh s}$$

$$E_z = \frac{q}{4\pi\epsilon_0 c^3}\,\Delta z\; 4(1.05180)$$

⇒ note it is independent of "a" since the walls are far away.

c)

Motion of charge.

$$m\ddot{z} = q \frac{dE_z}{d\Delta z} z \qquad z \sim e^{\gamma t}$$

$$\cancel{\boxed{A^2 = \frac{q^2}{4\pi\epsilon_0 c^3 m}\, 4(1.05180)}}$$

⇒ the motion of the particle away from the midplane is exponential

$$\boxed{\gamma^2 = \frac{q}{m}\frac{dE_z}{d\Delta z}}$$

⇒ particle acceleration will continue to increase as q approaches the wall and is closer to the image charge.