Hwk #1 Solutions

1.4 $\underset{\sim}{E} = E\hat{r}$ by symmetry

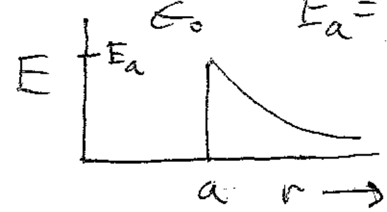
conducting sphere:

charge on surface

$r < a \Rightarrow \boxed{\underset{\sim}{E} = 0}$ since $\int_A \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{1}{\epsilon_0} \int dx\, e$
$= 0$

$r > a \qquad \int \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi r^2 E = \frac{Q}{\epsilon_0} \qquad E_a = \frac{Q}{4\pi\epsilon_0 a^2}$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$$



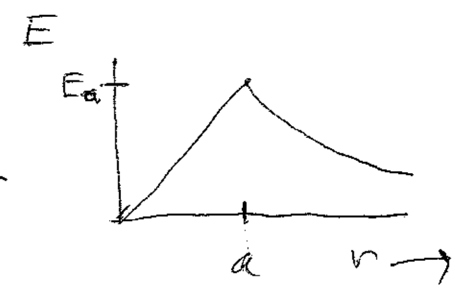
uniform charge:

$r > a$ same as above

$r < a \qquad \rho = \frac{Q}{\frac{4}{3}\pi a^3}$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \rho \frac{4}{3}\pi r^3 = \frac{Q}{\epsilon_0}\left(\frac{r}{a}\right)^3$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3}}$$



non-uniform charge:

$\rho = cQ\left(\frac{r}{a}\right)^n \frac{1}{a^3} \qquad c = \text{a constant}$

$$Q = \int_0^a 4\pi r^2 dr\, \rho = 4\pi \int_0^a \frac{r^2 dr}{a^3} cQ \frac{r^n}{a^n}$$

$$1 = 4\pi c \frac{a^{n+3}}{n+3} \frac{1}{a^{n+3}}$$

$$c = \frac{n+3}{4\pi}$$

$$\rho = \frac{Q}{\frac{4\pi}{n+3} a^3} \left(\frac{r}{a}\right)^n$$

$\cdot\, r > a$ same as above

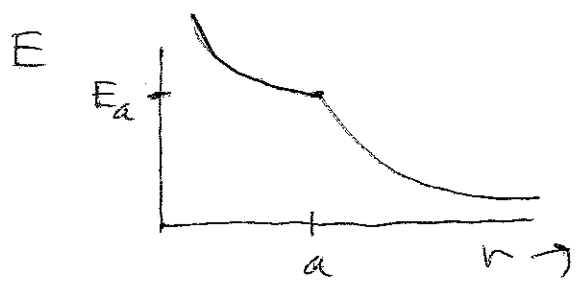
$r < a$

$$4\pi r^2 E = \frac{Q\ \cancel{4\pi}}{\frac{\cancel{4\pi}}{n+3}\epsilon_0} \int_0^r dr\, \frac{r^2}{a^3} \frac{r^n}{a^n}$$

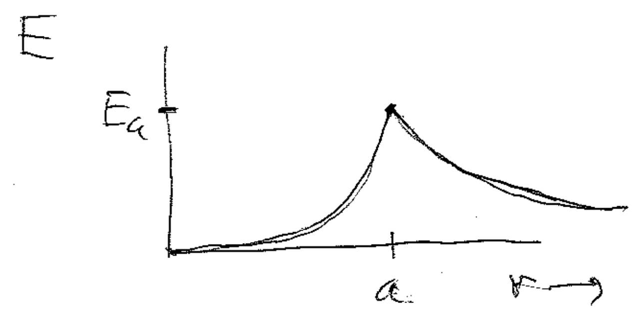
$$= \frac{\cancel{(n+3)}\,Q}{\epsilon_0} \frac{1}{a^{n+3}} \frac{r^{n+3}}{\cancel{n+3}}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0} \frac{r^{n+1}}{a^{n+3}}}$$

$n = -2$

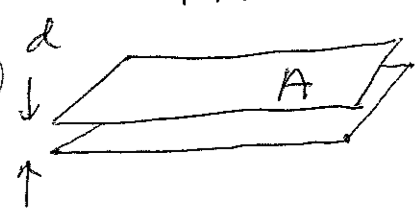


$n = 2$



1.6 Plates

a

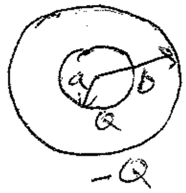


$$E = \frac{\sigma}{\epsilon_0}$$

$$V = Ed = \frac{\sigma}{\epsilon_0} d = Q \frac{d}{A\epsilon_0}$$

$$\boxed{C = \frac{A\epsilon_0}{d}}$$

(b) spheres



$$E\,4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$$

$$V = \frac{Q}{4\pi\epsilon_0}\int_a^b dr\,\frac{1}{r^2}$$

$$= \frac{Q}{4\pi\epsilon_0}\; -\frac{1}{r}\Big|_a^b$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\boxed{C = \frac{4\pi\epsilon_0\; ab}{b-a}}$$

(c) cylinders.

$$E\,2\pi r L = \frac{Q}{\epsilon_0}$$

$$\cancel{E = Q}$$

$$E = \frac{Q}{2\pi L\epsilon_0}\frac{1}{r}$$

$$\boxed{V = \frac{Q}{2\pi L\epsilon_0}\ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi L\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

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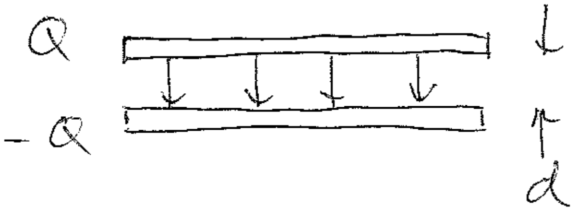
1.9 Find the attractive force between the plates of the parallel plate capacitor



a) fixed charges

on upper plate $E = \frac{\sigma}{\epsilon_0}$

but E_{ext} due to $-Q$ is only $E_{ext} = \frac{\sigma}{2\epsilon_0}$

$$\Rightarrow F = Q E_{ext} = \boxed{\frac{Q^2}{2A\epsilon_0}}$$

Using energy argument

$$W = \frac{1}{2}\epsilon_0 E^2 A d = \frac{1}{2} \cancel{\epsilon_0} A d \frac{\sigma^2}{\epsilon_0^{\cancel{2}}}$$

$$F = -\frac{\partial W}{\partial d} = \frac{1}{2}\frac{Q^2}{A\epsilon_0} \quad \text{downward}$$

b) fixed potential

same as before but how is the calculation done using the energy argument

$$W = \frac{1}{2}\epsilon_0 A \frac{V^2}{d} - \underbrace{QV}_{\text{work done by battery}}$$

$$= \frac{1}{2}\epsilon_0 A \frac{V^2}{d} - V^2\frac{A\epsilon_0}{d} = -\frac{1}{2}\frac{V^2 A\epsilon_0}{d}$$

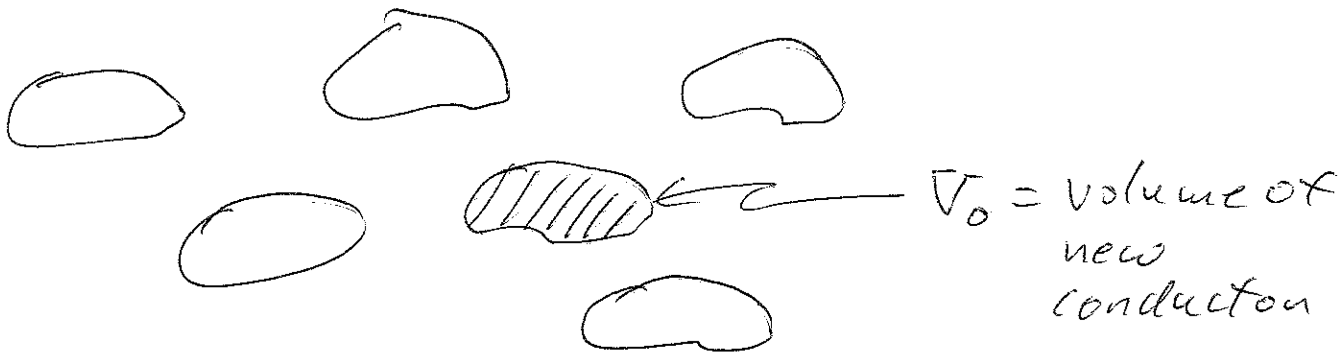
$$F = -\frac{\partial W}{\partial d} = \frac{1}{2}\frac{V^2 A\epsilon_0}{d^2}$$

$= \text{same}$

$\Rightarrow$ again attractive since $W \downarrow$ as $d \downarrow$

1.16

Consider fixed conductors with fixed total charge. Add an uncharged conductor (insulated). Show that stored energy is reduced.



Let φ_i = initial potential

φ_f = final potential $= \varphi_i + \delta\varphi$

$$W_i = \frac{\epsilon_0}{2}\int_V d\underset{\sim}{x}\ \nabla\varphi_i \cdot \nabla\varphi_i$$

$$W_f = \frac{\epsilon_0}{2}\int_{V-V_0} d\underset{\sim}{x}\ \nabla(\varphi_i+\delta\varphi)\cdot\nabla(\varphi_i+\delta\varphi)$$

since $\underset{\sim}{E}_f = 0$ in V_0

$$\Delta W = W_f - W_i$$

$$= \frac{\epsilon_0}{2}\int_{V-V_0} d\underset{\sim}{x}\left[\nabla\varphi_i\cdot\nabla\varphi_i + 2\nabla\varphi_i\cdot\nabla\delta\varphi + |\nabla\delta\varphi|^2\right]$$

$$-\frac{\epsilon_0}{2}\int_V d\underset{\sim}{x}\ \nabla\varphi_i\cdot\nabla\varphi_i$$

$$= -\frac{\epsilon_0}{2}\int_{V_0} d\underset{\sim}{x}\ |\nabla\varphi_i|^2 + \frac{\epsilon_0}{2}\int_{V-V_0} d\underset{\sim}{x}\left[2\nabla(\varphi_i+\delta\varphi-\delta\varphi)\cdot\nabla\delta\varphi + |\nabla\delta\varphi|^2\right]$$

$$= -\frac{\epsilon_0}{2}\int_{V_0} d\underset{\sim}{x}\, |\nabla \varphi_i|^2 - \frac{\epsilon_0}{2}\int_{V-V_0} d\underset{\sim}{x}\, |\nabla \delta\varphi|^2$$

$$\underbrace{+ \frac{\epsilon_0}{\cancel{2}}\cancel{2}\int_{V-V_0} d\underset{\sim}{x}\;\nabla(\varphi_i+\delta\varphi)\cdot\nabla\delta\varphi}_{I}$$

$$I = \int_{S_0} ds\,(\varphi_i+\delta\varphi)\frac{\partial}{\partial n}\delta\varphi - \int_{V-V_0} d\underset{\sim}{x}\,(\varphi_i+\delta\varphi)\underbrace{\nabla^2\delta\varphi}_{-\frac{\delta\rho}{\epsilon_0}}$$

$\varphi_i + \delta\varphi$ is constant on and within conductors.

$$I = \varphi_f \int_{S_0} ds\,\cancel{\left(-\frac{\delta\sigma}{\epsilon_0}\right)}^{\,0} + \cancel{\left[\int_{V-V_0} d\underset{\sim}{x}\,\frac{\delta\rho}{\epsilon_0}\right]}^{\,0}\varphi_f$$

since no net charge.

since total charge unchanged

$$= 0$$

$$\Delta W = -\frac{\epsilon_0}{2}\left[\int_{V_0} d\underset{\sim}{x}\,|\nabla\varphi_i|^2 + \int_{V-V_0} d\underset{\sim}{x}\,|\nabla\delta\varphi|^2\right]$$

but in V_0, $\underset{\sim}{E}_f = 0$ so $\nabla\varphi_i = -\nabla\delta\varphi$

$$\boxed{\Delta W = -\frac{\epsilon_0}{2}\int_V d\underset{\sim}{x}\,|\nabla\delta\varphi|^2 < 0}$$