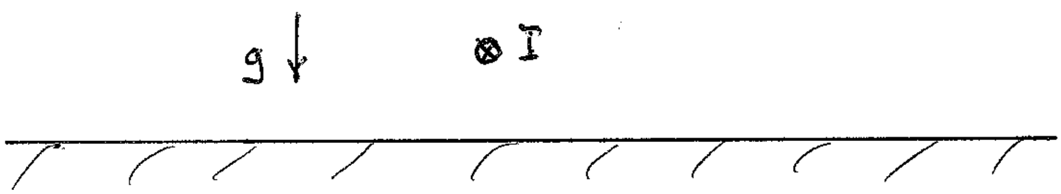
Physics 606 | Final Exam | Spring '12
Dr. Drake

1. (60 points) The following are short answer questions which should not require extensive calculations.

   15 (a) An electromagnetic wave with electric field amplitude E_0 is normally incident on a perfect conductor. Using the momentum per unit volume of the wave, calculate the force acting on the conductor.

   15 (b) An infinite wire carrying a current I and mass-per-unit-length μ is brought from infinity and comes to an equilibrium a distance d above a perfect conductor in a gravitational field g. Calculate a relation between I, g and d and whatever other universal constants are needed.



   15 (c) A TE mode of frequency ω propagates down a hollow cylindrical ideal conductor of radius a. Sketch the electric and magnetic fields for the lowest order mode. Estimate the minimum frequency at which energy will propagate down the system.

   15 (d) Consider an infinite sheet of charge per unit area σ. A dielectric of uniform thickness d is placed on top of the charged sheet. What is the induced charge on the dielectric? What is the net force between the dielectric and the charged sheet?

2. (~~50~~ 70 points) Consider an infinite dielectric rod with charge per unit length λ and radius "a". The rod is aligned with the z axis. The rod oscillates along the z direction with an amplitude d and frequency ω. The displacement Δz is given by $\Delta z = de^{-i\omega t}$. Assume that the charge lies on the surface of the rod.

   10 (a) Sketch the electric field in the $z - \rho$ plane. What is the direction of the magnetic field? The Poynting flux?

   10 (b) In the region $\omega\rho/c \gg 1$ what are the relative magnitudes of the oscillatory components of $\mathbf{E}$ and $\mathbf{B}$? How do they fall off with distance ρ in the same region?

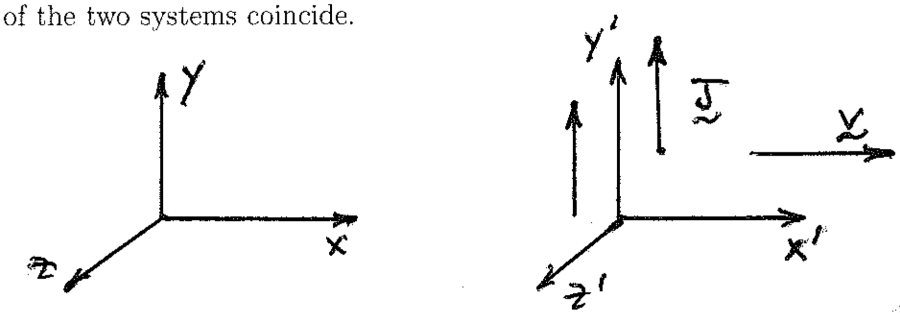
15 (c) Derive an equation for the vector potential A_z produced by the oscillating current J_z.

15 (d) Solve this equation for A_z in the region $\rho > a$. What is the boundary condition on the wave in this region? Solve for A_z in the region $\rho < a$. Match the solutions at $\rho = a$.

10 (e) Calculate $\mathbf{E}$ and $\mathbf{B}$ in the region $\omega\rho/c \gg 1$.

10 (f) Calculate the Poynting flux in this same region. Show that the total radiated power is independent of ρ.

3. (70 points) An infinite current sheet of current density $\mathbf{J}'(x') = K\delta(x')\hat{y}$ is at rest in the S' system. The S' system is moving along the positive x axis with a uniform velocity v with respect to the laboratory system S. At $t = t' = 0$ the origins of the two systems coincide.



15 (a) Calculate $\mathbf{E}'$ and $\mathbf{B}'$ in the rest frame of the current sheet.

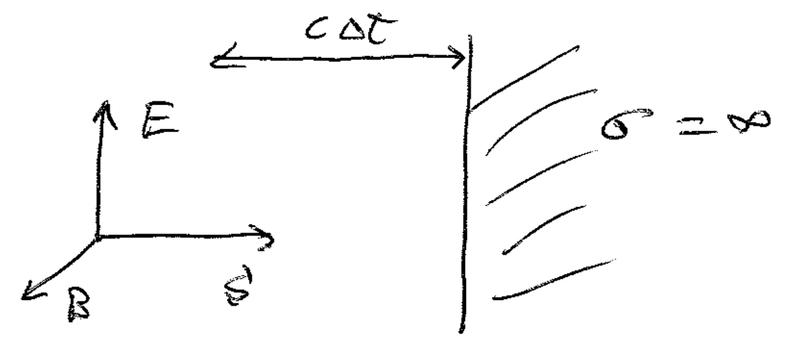
15 (b) Calculate the charge $\rho(x,t)$ and current $\mathbf{J}(x,t)$ densities as seen in the lab frame S. Express your answer in terms of the laboratory space/time coordinates.

20 (c) Calculate $\mathbf{E}$ and $\mathbf{B}$ as seen by the observer in the lab frame by transforming $\mathbf{E}'$ and $\mathbf{B}'$ directly. Again express your answer in the lab space/time coordinates.

20 (d) Calculate $\mathbf{E}$ and $\mathbf{B}$ directly in the lab frame by solving Maxwell's equations with ρ and $\mathbf{J}$ as sources.
(Hint: in the lab frame $\partial/\partial t$ and $\partial/\partial x$ are related. Use this to simplify the equations.)

Physics 606 Final Exam Solutions

1) a)



$$\frac{\text{momentum}}{\text{vol}} = \frac{\underset{\sim}{E_0} \times \underset{\sim}{H_0}^*}{2c^2} \qquad E_0 = cB_0 = c\mu_0 H_0$$

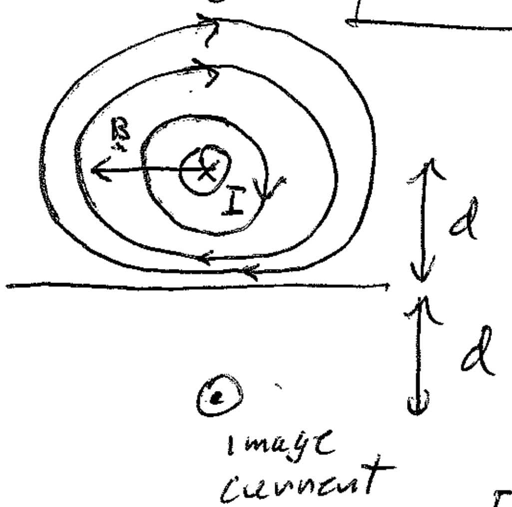
during a time Δt total momentum incident is

$$\Delta p = c\Delta t \, \frac{E_0^2}{2c^3\mu_0} = \frac{1}{2}\epsilon_0 E_0^2 \Delta t$$

$$F = \frac{2\Delta p}{\Delta t} = \boxed{\epsilon_0 E_0^2}$$

$\Rightarrow$ includes momentum of reflected wave

b)



B at I due to image current

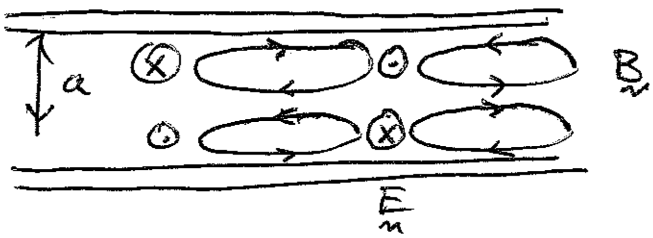
$$B = \frac{\mu_0 I}{2\pi(2d)}$$

$$\frac{F}{l} = IB = \mu g$$

$$\Rightarrow \boxed{\frac{\mu_0 I^2}{4\pi d} = \mu g}$$

TE mode $\Rightarrow E_z = 0,\ B_z \neq 0$

c)



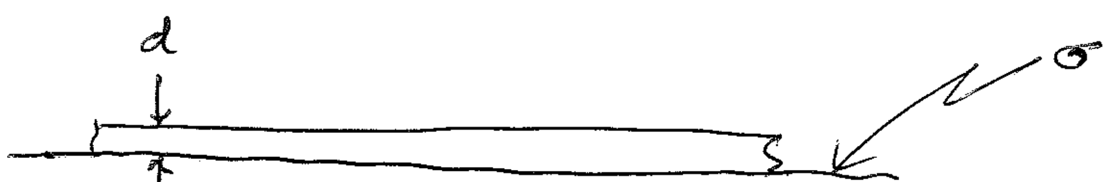
$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) B_z = 0$$

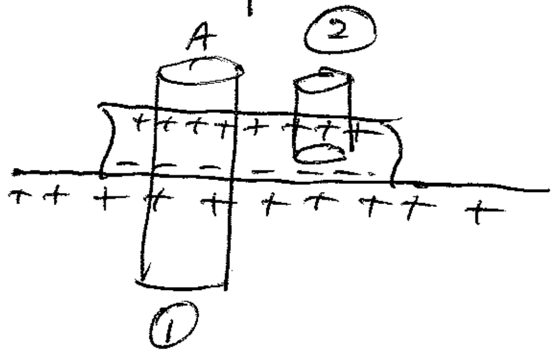
$$\nabla^2 = \nabla_t^2 - K_z^2 \sim -\frac{1}{a^2} - k_z^2 = -\frac{\omega^2}{c^2}$$

$$\Rightarrow k_z^2 = \frac{\omega^2}{c^2} - \frac{1}{a^2} > 0$$

$$\Rightarrow \omega^2 > \frac{c^2}{a^2}$$

d)





dielectric is polarized

E_0 outside dielectric use Gaussian surface ①

$$2AE_0 = \frac{\sigma}{\epsilon_0} A \qquad E_0 = \frac{\sigma}{2\epsilon_0}$$

Use Gaussian surface ② to calculate E_i inside dielectric. Since no free charge

$$D_0 = D_i = \epsilon_0 E_0 = \epsilon_i E_i$$

$$\Rightarrow E_i = \frac{\epsilon_0}{\epsilon_i} E_0 < E_0$$

But $\nabla \cdot \underset{\sim}{E} = \frac{\rho_{tot}}{\epsilon_0}$ For ② $\rho_{tot} = \rho_p$ = pol charge

$$E_0 - E_i = \frac{1}{\epsilon_0}\sigma_{pol} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{\epsilon_0}{\epsilon_i}\right)$$

$$\sigma_{pol} = \frac{\sigma}{2}\left(1 - \frac{\epsilon_0}{\epsilon_i}\right) \Rightarrow \text{top of dielectric}$$

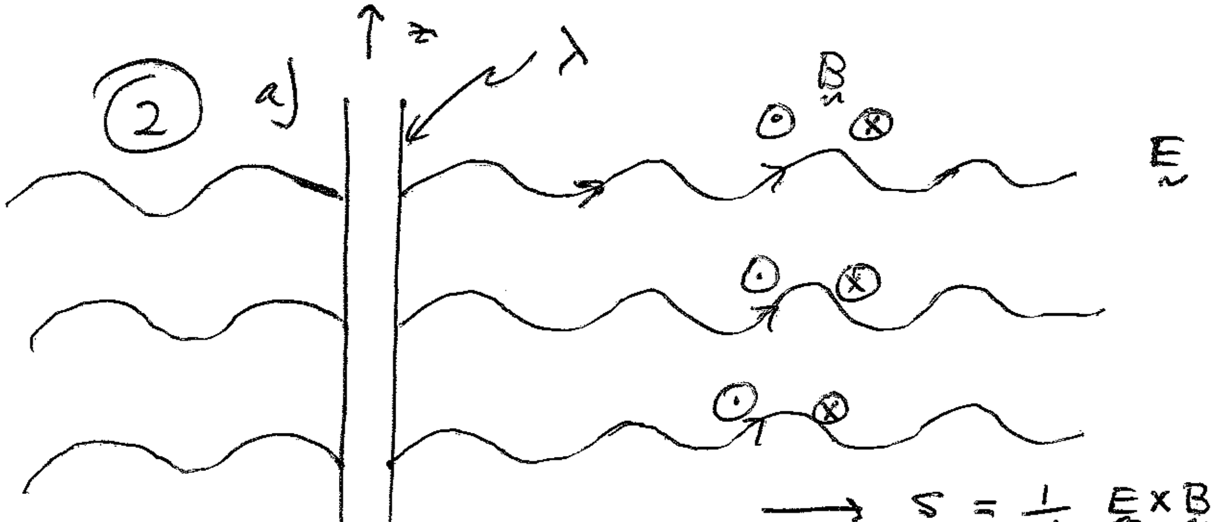
$$\Rightarrow -\sigma_{pol} \Rightarrow \text{bottom of dielectric}$$

Calculate force on σ_{pol} and $-\sigma_{pol}$ due to σ.

$\Rightarrow$ E due to σ is uniform so forces on σ_{pol} and $-\sigma_{pol}$ cancel

$\Rightarrow$ no net force

2 a)



$\longrightarrow \; \underset{\sim}{S} = \frac{1}{\mu_0} \underset{\sim}{E} \times \underset{\sim}{B}$

$\underset{\sim}{E}$ is along z
$\underset{\sim}{B}$ is in φ direction
$\vec{S}$ is radial

b) $\Rightarrow$ electromagnetic wave

$$E_z = -c B_\varphi$$

total power from radiation field is independent of ρ

$$\Rightarrow 2\pi\rho \vec{S} \sim \rho E^2 \sim \text{const}$$

$$E, B \sim \frac{1}{\rho^{1/2}}$$

c) $$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}\, \mu_0$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \qquad \underset{\sim}{J} = J_z \hat{z}$$

$$\underset{\sim}{E} = -\frac{\partial}{\partial t} \underset{\sim}{A} - \nabla \varphi \Rightarrow E_z = -\frac{\partial}{\partial t} A_z$$

$$[\nabla \times (\nabla \times \underset{\sim}{A})]_z = [\nabla\, \nabla \cdot \underset{\sim}{A} - \nabla^2 \underset{\sim}{A}]_z$$

$$= \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_z - \nabla^2 A_z$$

$$\frac{\partial}{\partial z} = 0$$

$$-\nabla^2 A_z = \mu_0 J_z - \cancel{\mu_0 \epsilon_0}\, \ddot{A}_z \frac{1}{c^2}$$

$$\nabla^2 A_z + \frac{\omega^2}{c^2} A_z = \mu_0 J_z$$

$$J_z = \rho v_z = -i\omega d \lambda\, \delta(\rho - a) \frac{1}{2\pi a}$$

$$\frac{1}{l} \int d\underset{\sim}{x}\, \rho = \lambda = \frac{l\, 2\pi}{l} \int_0^\infty d\rho\, \rho\, \rho$$

$$\Rightarrow \rho = \frac{\lambda}{2\pi a} \delta(\rho - a)$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} A_z + \frac{\omega^2}{c^2} A_z = \mu_0 \left(-\frac{i\omega d \lambda}{2\pi a} \delta(\rho - a) \right)$$

$$\boxed{A_z'' + \frac{1}{\rho} A_z' + \frac{\omega^2}{c^2} A_z = S_0\, \delta(\rho - a)}$$

$\Rightarrow$ Bessel Eqn with $\nu = 0$, $k = \frac{\omega}{c}$

$$S_0 = -\frac{i\omega d \lambda \mu_0}{2\pi a}$$

d) $\rho > a$

$$A_z'' + \frac{1}{\rho} A_z' + \frac{\omega^2}{c^2} A_z = 0$$

B.C. $\Rightarrow$ outgoing wave

$$A_z \sim H_0^{(1)} \sim \frac{e^{ik\rho}}{\rho^{1/2}} e^{-i\omega t} \Rightarrow \text{outgoing}$$

$$A_z = C^+ H_0^{(1)}\left(\frac{\omega}{c}\rho\right)$$

$\rho < a$

$\Rightarrow$ bounded at $\rho = 0$

$\Rightarrow J_0(k\rho)$

$$A_z = C^- J_0\left(\frac{\omega}{c}\rho\right)$$

Integration eqn for A_z across $\rho = a$

$$\left.\frac{\partial A_z}{\partial \rho}\right|_{a-\epsilon}^{a+\epsilon} = S_0'$$

$$\left. A_z \right|_{a-\epsilon}^{a+\epsilon} = 0 \Rightarrow C^+ = C^0 J_0\left(\frac{\omega}{c}a\right)$$

$$C^- = H_0^{(1)}\left(\frac{\omega}{c}a\right) C^0$$

$$C^0 \frac{\omega}{c}\left[H_0^{(1)\prime} J_0\left(\frac{\omega}{c}a\right) - H_0^{(1)}\left(\frac{\omega}{c}a\right) J_0'\left(\frac{\omega}{c}a\right)\right] = S_0'$$

$\cancel{A}$

$\underline{\rho > a}$ $\quad A_z = c^0 \, J_0\left(\frac{\omega}{c}a\right) H_0^{(1)}\left(\frac{\omega}{c}\rho\right)$

$\underline{\rho < a}$ $\quad A_z = c^0 \, J_0\left(\frac{\omega}{c}\rho\right) H_0^{(1)}\left(\frac{\omega}{c}a\right)$

$$c^0 = -\frac{i \cancel{\alpha} d \lambda \mu_0 c}{2\pi a \cancel{\alpha}} \left[H_0^{(1)\prime} J_0 - H_0^{(1)} J_0^{\prime} \right]^{-1}_{\rho = a}$$

e)

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$$B_\varphi = -\frac{\partial}{\partial \rho} A_z$$

$$A_z \simeq c^0 \, J_0\left(\frac{\omega}{c}a\right) \sqrt{\frac{2}{\pi \frac{\omega}{c}\rho}} \, e^{i\frac{\omega}{c}\rho - i\frac{\pi}{4}}$$

$$B_\varphi = -i\frac{\omega}{c} c^0 \, J_0\left(\frac{\omega}{c}a\right) \sqrt{\frac{2}{\pi \frac{\omega}{c}\rho}} \, e^{i\frac{\omega}{c}\rho - i\frac{\pi}{4}}$$

$$\cancel{B} \; E_z = -c B_\varphi$$

f)

$$S_\rho = -\frac{1}{2}\frac{1}{\mu_0} \cancel{\;} \operatorname{Re}\left(E_z B_\varphi^*\right)$$

$$= \frac{1}{2\mu_0} c \frac{\omega^2}{c^2} |c^0|^2 J_0^2\left(\frac{\omega}{c}a\right) \frac{2}{\pi \frac{\omega}{c}\rho}$$

total power

$$P \sim S_\rho \, 2\pi\rho \implies \text{indep of } \rho$$

3

a) $\underset{\sim}{J}'(x') = K\,\delta(x')\,\hat{y}$

$\nabla \times \underset{\sim}{B}' = \mu_0 K\,\delta(x')\,\hat{y}$

$-\frac{\partial}{\partial x} B_z' = \mu_0 K\,\delta(x')$

$$\boxed{B_z' = -\frac{\mu_0 K}{2} H(x')}$$

define
$H = 1 \quad x' > 0$
$\;\;\, = -1 \quad x' < 0$

$\rho' = 0$ and $\frac{\partial}{\partial t'} = 0 \Rightarrow \boxed{\underset{\sim}{E}' = 0}$

b) Lab frame S' $\Rightarrow (c\rho, J)$ are a 4-vector

$c\rho = \gamma\left(c\rho' + \beta J_x'\right) = 0$

$\cancel{\sigma_z = B_z'}$ $\quad J_y = J_y'$

$$\boxed{J_y = K\,\delta(\gamma(x - \beta c t))}$$

$x' = \gamma(x - \beta c t)$

$$\boxed{\rho = 0}$$

c) $\frac{1}{c}E_y = \gamma\left[\cancel{\frac{1}{c}E_y'}^{\,0} - \underset{\sim}{\beta} \times \underset{\sim}{B}_1'\right]_y$

$= \gamma\beta B_z'$

$$\boxed{E_y = -c\,\frac{\gamma\beta\mu_0 K}{2}\,H(\gamma(x - \beta c t))}$$

$$B_z = \gamma\left(B_z' + \cancel{\underset{\sim}{\beta} \times \tfrac{1}{c} \underset{\sim}{E}_\perp'}^{\,0}\right)$$

$$\boxed{B_z = -\gamma \frac{\mu_0 K}{2} H(\gamma(x - \beta c t))} \quad \boxed{E_y = v B_z}$$

d) $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \underset{\sim}{E}$

$$-\frac{\partial}{\partial x} B_z = \mu_0 J_y + \frac{1}{c^2} \frac{\partial}{\partial t} E_y$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

$$\frac{\partial}{\partial x} E_y + \frac{\partial}{\partial t} B_z = 0$$

$$J_y = J_x(x - vt)$$

$$\Longrightarrow \frac{\partial}{\partial t} = -v \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} E_y - v \frac{\partial}{\partial x} B_z = 0$$

$$\boxed{E_y = v B_z} \quad \boxed{B_z = -\frac{\mu_0 \gamma K}{2} \otimes H(\gamma(x - vt))}$$

$$-\frac{\partial}{\partial x} B_z = \mu_0 J_y + \frac{1}{c^2}\left(-v \frac{\partial}{\partial x}\right) v B_z$$

$$\frac{\partial}{\partial x} B_z = -\mu_0 \gamma^2 J_y$$

$$\Delta B_z = -\mu_0 \gamma^2 K \int dx\, \delta(\gamma(x - vt))$$