Physics 604 Final Exam Fall '11

Dr. Drake

1. (40 pts)

   (a) Evaluate the following integral

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$$I = \int_{-\infty}^{\infty} dz \frac{1}{z^2}(1 - \cos z)$$

   (b) The function f is defined by

$$f(z) = \sqrt{z^2 - 1}$$

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   with a branch cut between -1 and 1. Give a plausibility argument why the branch cut does not need to be extended to infinity. Evaluate $f(ia)$ where a is real and positive.

2. (40 pts) The differential equation for the modified Bessel function is

$$z^2 y'' + zy' - (z^2 + \nu^2)y = 0$$

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   (a) What is the lowest order behavior of the solutions of this equation in the vicinity of $z = 0$? Consider all possible values of real ν.

   (b) An integral representation of the modified Bessel function of the first kind, $I_\nu(z)$ is

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$$I_\nu(z) = \frac{1}{2\pi i} \int_C e^{(z/2)(s+1/s)} \frac{ds}{s^{1+\nu}},$$

   where a cut extends from zero to negative infinity in the s plane and the contour C wraps around the cut, starting at negative infinity below the cut and ending at negative infinity above the cut. Evaluate the integral approximately for z large and positive.

3. (60 pts) Consider a two-dimensional cylindrical cavity of inner radius a and outer radius b that has an angular width ϕ_0. The electric potential V is maintained at $-V_0$ and V_0 at $\phi = -\phi_0/2$ and $\phi = \phi_0/2$, respectively, and zero at $r = a, b$. The potential $V(r, \phi)$ satisfies Poisson's equation

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} V + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} V = 0$$

   (a) It is convenient to change to a new variable $s = ln(r)$ before attempting to solve this problem. What is the equation satisfied by V when expressed in terms of s and ϕ?

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(b) Write the solution for V in terms of a separable set of eigenfunctions $\Phi_m(\phi)$ and $R_m(s)$ as follows:

$$V(s,\phi) = \sum_m c_m R_m(s)\Phi_m(\phi)$$

25 Write equations for $\Phi_m(\phi)$ and $R_m(s)$ and solve these equations using boundary conditions appropriate for the solution V. Normalize the basis functions so they have unit norm.
Hint: Use the symmetry in ϕ to simplify the solution.

20 (c) Solve for c_m by matching the form of the potential at the side boundaries.

4. (60 points) A liquid is heated in a hollow sphere of radius b by a pulse of heat at $t = 0$ at radius $r = r_0$. The equation satisfied by the liquid is

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = A\delta(t)\delta(r - r_0)$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

The temperature of the liquid is zero for $t < 0$ and remains zero at the boundary for all time.

25 (a) Construct a set of basis functions $\phi_n(r)$ to describe the liquid in the cavity. Write the basis functions as a linear combination of the two solutions of the Bessel equation $J_\nu(kr)$ and $Y_\nu(kr)$. Give expressions for the eigenvalues of your basis functions (in terms of the known properties of $J_\nu(kr)$ and $Y_\nu(kr)$) and define the normalization so that the $\phi_n(r)$ have unity norm. State why the basis functions are orthogonal (you don't have to prove orthogonality). What is the behavior of the eigenfunctions near $r = 0$? Sketch the lowest three eigenfunctions.
Hint: Bessel's equation is $r^2 y'' + r y' + (k^2 r^2 - \nu^2) y = 0$. Let $\phi_n = g_n / r^{1/2}$ and show that g_n satisfies Bessel's equation. What is ν?

15 (b) Write $T(r,t)$ as

$$T(r,t) = \sum_{n=1}^{\infty} C_n(t)\phi_n(r)$$

and derive an equation for $C_n(t)$. What is the characteristic damping rate of the eigenfunctions?

15 (c) Solve for $C_n(t)$ and then write a solution for the complete space/time dependence of T.

5 (d) What is the lowest order non-trivial form of the solution at late time?

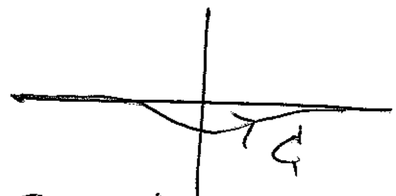
Final Exam Sollutions   Physics 604

1 a) Evaluate

$$I = \int_{-\infty}^{\infty} dz \frac{1}{z^2}\left(1-\cos z\right)$$

Note that there is no singularity at $z=0$
$\Rightarrow$ move the contour integral away from $z=0$ so can split $\cos(z)$ into exponential components

$$I = \int_C dz \frac{1}{z^2}\left(1-\cos z\right)$$



$$= \underbrace{\int_C dz \frac{1}{z^2}}_{I_1} - \underbrace{\frac{1}{2}\int_C dz \frac{e^{iz}}{z^2}}_{I_2} - \underbrace{\frac{1}{2}\int_C dz \frac{e^{-iz}}{z^2}}_{I_3}$$

Close I_1 at ∞ in LHP $\Rightarrow$ no contribution from semi-circle $\sim \frac{1}{R} \to 0$

$I_1 = 0$ ~~since~~ since no enclosed poles

Close I_2 in UHP $\Rightarrow$ no contribution from semi-circle $\Rightarrow$ Jordan's Lemma

$$I_2 = -\frac{1}{\cancel{2}}\cancel{2}\pi i \left(e^{iz}\right)'\Big|_{z=0} = \pi$$

Close I_2 in LHP $\Rightarrow$ no enclosed sing.
$\Rightarrow 0$

$$\boxed{I = \pi}$$

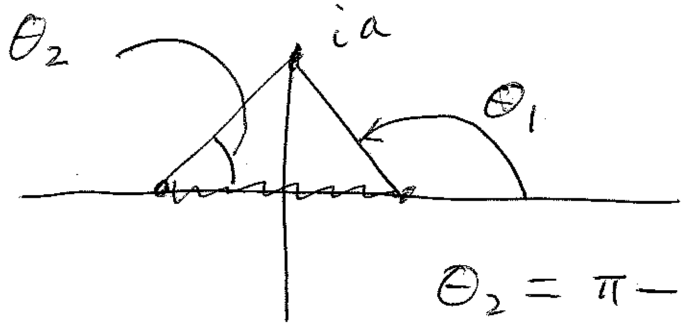
6)

$$f(z) = \sqrt{z^2-1}$$

At large z $\quad f \Rightarrow z$ which is a single valued function $\Rightarrow$ cut not needed at ∞.

Evaluate at $z = ia$

$$f(z) = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}$$



$$(z-1)^{\frac{1}{2}} = \sqrt{1+a^2}\, e^{\frac{i\theta_1}{2}}$$

$$(z+1)^{\frac{1}{2}} = \sqrt{1+a^2}\, e^{\frac{i\theta_2}{2}}$$

$$\theta_2 = \pi - \theta_1$$

$$f(ia) = (1+a^2)\, e^{i\frac{1}{2}(\cancel{\theta_1} + \pi - \cancel{\theta_1})}$$

$$\boxed{= (1+a^2)\, e^{i\frac{\pi}{2}}}$$

2 a) $z^2 y'' + z y' - (z^2 + \nu^2) y = 0$

near $z = 0$:

$$z^2 y'' + z y' - \nu^2 y = 0$$

$\Rightarrow$ Euler equ $\Rightarrow y \sim z^p$

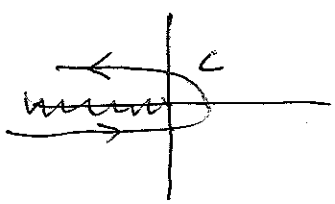
$$\cancel{z^2}\, p(p-1) \cancel{z^{p-2}} + p \cancel{z}\, \cancel{z^{p-1}} - \nu^2 \cancel{z^p} = 0$$

$$p^2 = \nu^2 \qquad p = \pm\nu$$

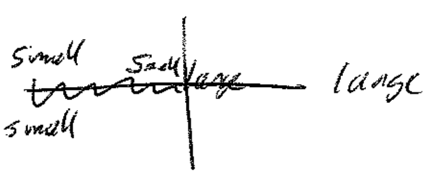
$$y \sim z^{\pm\nu}$$

for $\nu = 0$, $y \sim 1, \ln z$

b) $$I_\nu(z) = \frac{1}{2\pi i} \int_C \frac{ds}{s^{1+\nu}}\, e^{\frac{z}{2}\left(s + \frac{1}{s}\right)}$$



$\Rightarrow$ for large positive z the integrand is dominated by the exponential



$\Rightarrow$ look for the SPs

$$h \equiv \frac{z}{2}\left(s + \frac{1}{s}\right)$$

$$h' = \frac{z}{2}\left(1 - \frac{1}{s^2}\right) = 0 \Rightarrow s_{sp} = \pm 1$$

$$h'' = \frac{z}{2}\frac{2}{s^3}$$

near $s = 1$

$$e^h \simeq e^z \, e^{\frac{1}{2}z(s-1)^2} \Rightarrow \text{PSD} \quad s-1 \sim re^{i\frac{\pi}{2}}$$

near $s = -1$

$$e^h \simeq e^{-z} \, e^{-\frac{1}{2}z(s+1)^2} \Rightarrow \text{PSD} \quad s+1 \sim r$$

$\Rightarrow$ small

Dominant contribution from SP at $s = 1$

$$I_\nu \simeq \frac{1}{2\pi \cancel{i}} e^z \int_{-\infty}^{\infty} dr \, \cancel{e^{i\frac{\pi}{2}}} \, e^{-\frac{1}{2}zr^2}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2}{z}}\sqrt{\pi}\, e^z \quad \boxed{= \frac{1}{\sqrt{2\pi z}} e^z}$$

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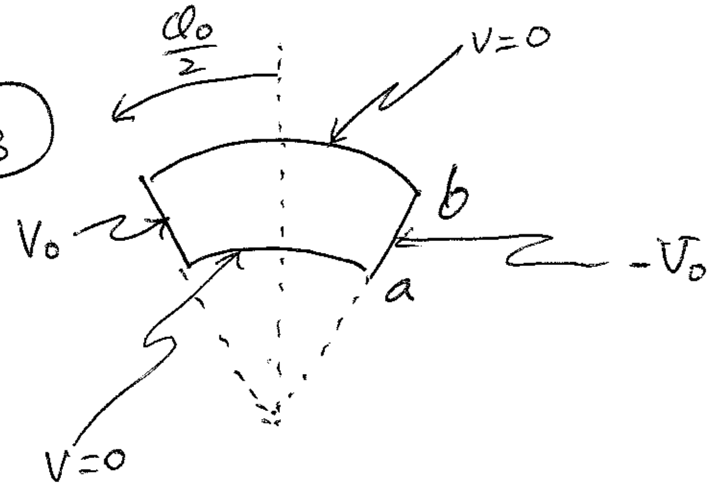

a) Let $s = \ln(r)$

$$ds = \frac{1}{r}\,dr$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial^2}{\partial s^2}V + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}V = 0$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial s^2}V + \frac{\partial^2}{\partial \varphi^2}V = 0}$$

b) $$V = \sum_m R_m(s)\,\Phi_m(\varphi)$$

Want oscillatory functions in s since need to match V at $\pm\frac{\varphi_0}{2}$.

$$= \sum_m \left(\frac{\partial^2 R_m}{\partial s^2}\Phi_m + R_m\frac{\partial^2 \Phi}{\partial \varphi^2}\right) = 0$$

$$= \sum_m R_m\Phi_m\left(\frac{1}{R_m}\frac{\partial^2 R}{\partial s^2} + \frac{1}{\Phi_m}\frac{\partial^2 \Phi_m}{\partial \varphi^2}\right) = 0$$

Let $$\frac{\partial^2 R_m}{\partial s^2} \equiv -k_m^2 R_m$$

$R_m \sim \sin k_m s, \cos k_m s$

Want ~~...~~ $R_m = 0$ at $s = \ln(a), \ln(b)$

$$\boxed{R_m = N_m \sin k_m(s - \ln a)} \Rightarrow 0 \text{ at } s = \ln a$$

at $s = \ln b$ must have $R_m = 0$

$$k_m(\ln b - \ln a) = m\pi$$

$$\boxed{k_m = \frac{m\pi}{\ln(b/a)}}$$

Normalization

$$\int_{\ln a}^{\ln b} ds\, N_m^2 \sin^2(\ ) = 1$$

$$N_m^2 \frac{1}{2} \ln\left(\frac{b}{a}\right) = 1$$

$$\boxed{N_m = \left(\frac{2}{\ln b/a}\right)^{\frac{1}{2}}}$$

$$\frac{\partial^2 \Phi_m}{\partial \ell^2} = k_m^2 \Phi_m$$

$$\Phi_m \sim \sinh(k_m \ell), \cosh(k_m \ell)$$

$\Rightarrow$ solution is anti-symmetric across $\ell = 0$

$$\Rightarrow \boxed{\Phi_m \sim \sinh(k_m \ell)}$$

c) Match at $\ell = \frac{\ell_0}{2}$ ($-\frac{\ell_0}{2}$ automatically ok)

$$V_0 = \sum_m c_m R_m(s) \sinh\left(k_m \frac{\ell_0}{2}\right)$$

$\Rightarrow$ multiply by $R_n(s)$ and integrate over s

$$V_0 = \int_{\ln a}^{\ln b} ds\; N_n \sin k_n (s - \ln a) = C_n \sinh\left(\frac{k_n \ell_0}{2}\right)$$

$$\parallel$$

$$- N_n \left. \frac{\cos k_n (s - \ln a)}{k_n} \right|_{\ln a}^{\ln b}$$

$$\parallel$$

$$N_n \frac{-\cos n\pi + 1}{k_n} = \begin{cases} 0 & n \text{ even} \\ \dfrac{N_n 2}{k_n} & n \text{ odd} \end{cases}$$

$$\boxed{C_n = \frac{2 N_n}{k_n} \frac{1}{\sinh \frac{k_n \ell_0}{2}} \quad \text{for } n \text{ odd}}$$

4 8

a) Construct basis functions

$$T = \sum_n c_n(t)\,\phi_n(r)$$

choose

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\phi_n + k_n^2 \phi_n = 0$$

S-L form

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\phi_n + k_n^2 r^2 \phi_n = 0$$

$$\Rightarrow P(r) = r^2, \quad w = r^2$$

$$r^2 \phi_n'' + 2r\phi_n' + k_n^2 r^2 \phi_n = 0$$

Let $\phi_n = \frac{g_n}{r^p}$ to match Bessel eqn for g_n

$$r^2\left(\frac{g_n''}{\cancel{r^p}} - \frac{2p g_n'}{r^{\cancel{p}+1}} + \frac{g_n p(p+1)}{r^{\cancel{p}+2}}\right)$$

$$+ 2r\left(\frac{g_n'}{\cancel{r^p}} - \frac{g_n p}{r^{\cancel{p}+1}}\right) + k_n^2 r^2 \frac{g_n}{\cancel{r^p}} = 0$$

want $\left(-2p\cancel{r} + 2\cancel{r}\right) g_n' = \cancel{r}\, g_n'$

$$2 - 2p = 1 \quad \Rightarrow p = \frac{1}{2}$$

$$a_n = \frac{g_n}{r^{1/2}}$$

$$r^2 g_n'' + r g_n' + g_n\left(k_n^2 r^2 - 1 + \tfrac{3}{4}\right) = 0$$

$$r^2 g_n'' + r g_n' + \left(k_n^2 r^2 - \tfrac{1}{4}\right) g_n = 0$$

$$\nu = \tfrac{1}{2}$$

$$\boxed{a_n = N_n \frac{J_{\frac{1}{2}}(k_n r)}{r^{1/2}}}$$ since $Y_{\frac{1}{2}} \to \infty$ at $r=0$.

want $a_n = 0$ at $r = b$

$k_n b = X_{\frac{1}{2}n}$ where $J_{\frac{1}{2}}(X_{\frac{1}{2}n}) = 0$

$$\boxed{k_n = \frac{X_{\frac{1}{2}n}}{b}}$$

Behavior near $r = 0$

$r^2 a_n'' + 2r a_n' \simeq 0 \Rightarrow$ Euler

$a_n \sim r^q$

$q(q-1) + 2q = 0$

$q(q+1) = 0 \qquad q = 0, -1$

$a_n \sim r^0, r^{-1}$

BC.

$$r^2 \mathcal{R}_n \mathcal{R}_m' \Big|_0^b = 0$$

$\Rightarrow$ satisfied at 0, b

$\Rightarrow$ $\mathcal{R}_n(r)$ are orthogonal because satisfy S-L eqn with BCs.

$\Rightarrow$ normalization $N_n^2 \int_0^b dr\, r\, J_{\frac{1}{2}}^2\left(x_{\frac{1}{2}n} \frac{r}{b}\right) = \boxed{N_n^2 \frac{b^2}{2} J_{\frac{3}{2}}^2(x_{\frac{1}{2}n}) = 1}$

b) Insert $T(r,t)$ into eqn.

$$\sum_{n=1}^{\infty} \left(\dot{c}_n + \kappa k_n^2 c_n\right) \mathcal{R}_n = A\, \delta(t)\, \delta(r - r_0)$$

multiply by $\mathcal{R}_m\, w$ and integrate

$$\dot{c}_m + \kappa k_m^2 c_m = A\, \delta(t)\, r_0^2\, \mathcal{R}_m(r_0)$$

damping rate $\boxed{-\kappa k_m^2}$

c) For $t \neq 0$, $c_m \sim e^{-\kappa k_m^2 t}$

$t < 0$, $c_m = 0$

$t > 0$, $c_m = c_m(0)\, e^{-\kappa k_m^2 t}$

jump near $t = 0$ $\quad c_m(0) = A r_0^2 \mathcal{R}_m(r_0)$

$$\boxed{c_m(t) = A r_0^2 \mathcal{R}_m(r_0)\, e^{-\kappa k_m^2 t}}$$

$$T = \sum_{n=1}^{\infty} \varphi_n(r)\, A r_0^2 \varphi_n(r_0)\, e^{-\kappa k_n^2 t}$$

$$\varphi_n(r) = \frac{1}{r^{1/2}} J_{\frac{1}{2}}(k_n r)$$

$$k_n = \frac{x_{\frac{1}{2}n}}{6}, \quad N_n = \frac{\sqrt{2}}{6} J_{\frac{3}{2}}(x_{\frac{1}{2}n})$$

d) Late time $n=1$ dominates

$$\boxed{T \simeq \varphi_1(r)\, A r_0^2 \varphi_1(r_0)\, e^{-\kappa k_1^2 t}}$$