Hwk # 9 Solutions

15.3.16

$$f(\underset{\sim}{r}) = \frac{1}{(2\pi)^{\cancel{3/2}\,3}} \int d\underset{\sim}{k}\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \frac{1}{k^2}$$

$$= \frac{1}{(2\pi)^{2}\,2} \cancel{2\pi} \int_0^\infty \frac{k^2\,dk}{k^2} \int_{-1}^{1} d(\cos\theta)\, e^{-ikr\cos\theta}$$

$d\underset{\sim}{k} = 2\pi k^2 dk\, d(\cos\theta)$

$\underset{\sim}{k}\cdot\underset{\sim}{r} = kr\cos\theta$

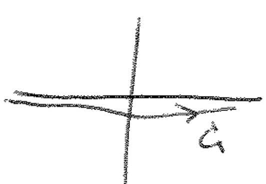
$$= \frac{1}{4\pi^2} \int_0^\infty dk\; \frac{e^{-ikr} - e^{ikr}}{-ikr}$$

$$= \frac{1}{4\pi^2 r}\, 2 \int_0^\infty \frac{dk}{k} \sin(kr)$$

$$= \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} dk\; \frac{1}{k} \sin(kr)$$

$\Rightarrow$ note: no singularity at $k=0$ so can move to contour below the origin

$$= \frac{1}{4\pi^2 r} \int_{C_1} dk\; \frac{1}{k}\, \frac{e^{ikr} - e^{-ikr}}{2i}$$



Integral with e^{ikr} can close above

$\Rightarrow$ gives residue at $k=0$

~~Int~~ Integral with e^{-ikr} close below

$\Rightarrow$ gives zero $\Rightarrow$ no sing.

$$= \frac{1}{4\pi^2 r}\,\frac{1}{2i}\, 2\pi i = \boxed{\frac{1}{4\pi r}}$$

15.3.17

a) Calculate FT of $f = e^{-a|x|}$

$$F(k) = \int_{-\infty}^{\infty} dx \, e^{-a|x|} e^{-ikx}$$

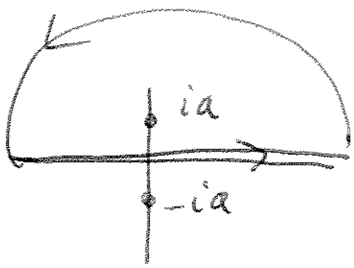
$$= \int_{0}^{\infty} dx \, e^{-ax} e^{-ikx} + \int_{-\infty}^{0} dx \, e^{ax} e^{-ikx}$$

$$= \frac{+1}{+a+ik} + \frac{1}{a-ik} = \boxed{\frac{2a}{a^2+k^2}}$$

b) Inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx} \frac{2a}{a^2+k^2}$$

for $x > 0$
close in UHP
$\Rightarrow$ residue at
$k = ia$



$$= \frac{1}{2\pi} 2a \frac{e^{-ax}}{2ia} 2\pi i = e^{-ax}$$

for $x < 0$
close in LHP
$\Rightarrow$ residue at
$k = -ia$

$$= \frac{1}{2\pi} 2a \frac{e^{ax}}{-2ia} (-2\pi i) = e^{ax}$$

$$\Rightarrow f(x) = e^{-a|x|}$$

15.4.3 Take FT of

$$\int_{-\infty}^{\infty} dx\, e^{-ikx} \left(-D\, \mathcal{C}'' + K^2 D\, \mathcal{C} = Q\, \delta(x) \right)$$

$$k^2 D\, \mathcal{C}(k) + K^2 D\, \mathcal{C}(k) = Q$$

$$\mathcal{C}(k) = \frac{Q \frac{1}{D}}{k^2 + K^2}$$

$$\mathcal{C}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{Q}{D}\, \frac{1}{k^2 + K^2}$$

$\Rightarrow$ same integral as 15.3.17 (b)

$$= \frac{Q}{2KD}\, e^{-K|x|}$$

2 a) Consider $\dot{T} - D T_{xx} = H(t)\, \delta(x)$

First do LT : $\int_0^{\infty} dt\, e^{i\omega t} (\ )$

$$-i\omega T(\omega) - \cancel{T(0,x)}^{\,0} - D\, T(\omega, x)_{xx} = \delta(x) \left(-\frac{1}{i\omega} \right)$$

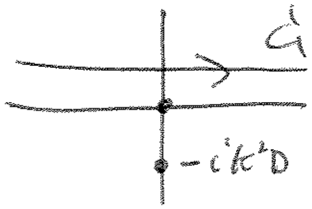
since source is zero for $t < 0$

Do FT : $\int_{-\infty}^{\infty} dx\, e^{-ikx} (\ )$

$$-i\omega\, T(k,\omega) - \cancel{T(0,k)}^{\,0} + k^2 D\, T(k,\omega) = -\frac{1}{i\omega}$$

$$T(k,\omega) = -\frac{1}{\omega}\, \frac{1}{\omega + ik^2 D}$$

$$T(k,t) = -\frac{1}{2\pi}\int_C d\omega\, e^{-i\omega t}\,\frac{1}{\omega}\,\frac{1}{\omega+ik^2D}$$



For $t>0$ close in LHP and pick up residues at $\omega=0,\,-ik^2D$.

$$T(k,t) = +\frac{\cancel{2\pi i}}{\cancel{2\pi}}\left(\frac{1}{\cancel{i}k^2D} + \frac{e^{-k^2Dt}}{-\cancel{i}k^2D}\right)$$

$$= +\frac{1}{k^2D}\left(1-e^{-k^2Dt}\right)$$

$$T(x,t) = \frac{1}{2\pi D}\int_{-\infty}^{\infty} dk\, e^{ikx}\left(\frac{1-e^{-k^2Dt}}{k^2}\right)$$

b)

$$T(0,t) = \frac{1}{2\pi D}\int_{-\infty}^{\infty} dk\,\frac{1-e^{-k^2Dt}}{k^2Dt}\,Dt$$

Let $s^2 = k^2Dt$

$$= \frac{1}{2\pi\cancel{D}}\,\frac{1}{(Dt)^{\frac{1}{2}}}\,\cancel{D}t\int_{-\infty}^{\infty} ds\left(\frac{1-e^{-s^2}}{s^2}\right)$$

$$= \frac{1}{2\pi D^{1/2}}\,t^{1/2}\int_{-\infty}^{\infty} ds\,\frac{1-e^{-s^2}}{s^2}$$

$$\sim \left(\frac{t}{D}\right)^{1/2} \Longrightarrow \text{no steady state}$$

For what range of x is this result valid?

Integrand in integral over k is peaked in the region $k^2 Dt \lesssim 1$ or

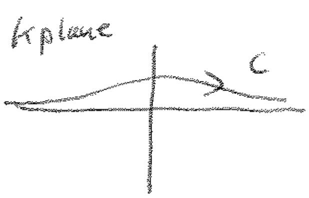
$$k^2 \lesssim \frac{1}{Dt}$$

Neglecting x is valid for $kx < 1$

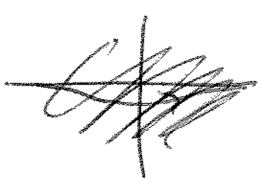
or $x < k^{-1} \sim (Dt)^{1/2}$

$$\Rightarrow \boxed{x^2 < Dt}$$

c) Consider x large and positive. Since no singularity at $k=0$ can shift contour off the real axis



$$T = \frac{1}{2\pi D} \int_{C_1} dk \left(1 - e^{-k^2 Dt}\right) \frac{e^{ikx}}{k^2}$$



$$\int_{C_1} dk \frac{e^{ikx}}{k^2} = 0$$ close in UHP $\Rightarrow$ no residue

$$T = -\frac{1}{2\pi D} \int_{C_1} dk\, e^{-k^2 Dt} \frac{e^{ikx}}{k^2}$$

Let $h = ikx - k^2 Dt$ $\quad h(k_{sp}) = -\frac{x^2}{4Dt}$

$h' = ix - 2kDt \quad \Rightarrow k_{sp} = \frac{ix}{2Dt}$

$h'' = -2Dt$

$$T \simeq -\frac{1}{2\pi D} \frac{1}{\left(\frac{ix}{2Dt}\right)^2} \int_{-\infty}^{\infty} ds\, e^{-Dts^2} e^{-\frac{x^2}{4Dt}}$$

$s = k - k_{sp}$

$$T \simeq + \frac{1}{2\pi \cancel{D}} \frac{4D^{\frac{1}{2}}t^2}{X^2} \frac{e^{-\frac{X^2}{4Dt}}}{(Dt)^{1/2}} \sqrt{\pi}$$

$$\boxed{T = \frac{2}{\sqrt{\pi}} \frac{D^{\frac{1}{2}} t^{3/2}}{X^2} e^{-\frac{X^2}{4Dt}}}$$

From the s integral have $k - k_{sp} \simeq \frac{1}{(Dt)^{1/2}}$

Want $k - k_{sp} \ll k_{sp}$

$$\frac{1}{(Dt)^{1/2}} \ll \frac{X}{2Dt} \Rightarrow \boxed{X^2 \gg 4Dt}$$

3 Consider

$$y_{xx} + k^2(x)\, y = 0$$

where $k^2 = 0$ at $x = \pm x_0$. $\Rightarrow$ k symmetric

a) Even and odd WKB solutions:

In general

$$y \sim \frac{e^{\pm i \int^x dx'\, k(x')}}{k^{1/2}}$$

Even:

$$y = C_e \frac{\cos\left[\int_0^x dx'\, k(x')\right]}{k^{1/2}}$$

$$y(-x) = C_e \frac{\cos\left[\int_0^{-x} dx'\, k\right]}{k^{1/2}} = \frac{C_e \cos\left[-\int_0^x dx'\, k\right]}{k^{1/2}}$$

$$= y(x)$$

Odd:

$$y = C_o \frac{\sin\left[\int_0^x dx'\, k(x')\right]}{k^{1/2}}$$

As before $y(-x) = -y(x)$

Note: must choose lower bound of x' integral as zero for ~~this to work~~ even/odd solutions

b) Evaluate WKB solutions close to TP at $x = x_0$

$$\Phi(x) \equiv \int_0^x dx'\, k(x') = \underbrace{\int_0^{x_0} dx'\, k}_{\Phi_0} + \int_{x_0}^x dx'\, k(x')$$

In second integral x' always close to x_0 so

$$k \simeq \left[-V'(x_0)(x'-x_0)\right]^{\frac{1}{2}}$$

$$= V_0'^{\,1/2} (x_0 - x')^{\frac{1}{2}}$$

$$\Phi(x) = \Phi_0 + V_0'^{\,1/2} \int_{x_0}^x dx' (x_0 - x')^{\frac{1}{2}}$$

$$= \Phi_0 - V_0'^{\,1/2} \frac{2}{3} (x_0 - x)^{3/2}$$

$$y_e = \frac{C_e}{k^{1/2}} \cos\left[\Phi_0 - \frac{2}{3}\sqrt{V_0'}\,(x_0 - x)^{3/2}\right]$$

$$y_o = \frac{C_o}{k^{1/2}} \sin\left[\quad\right]$$

c) Near $x = x_0$,

$$y_{xx} - V_0'(x-x_0)\,y = 0 \qquad \text{Let } t = \frac{x-x_0}{\Delta}$$

$$y_{tt} - V_0' \Delta^2 \Delta t\, y = 0$$

$$\Delta = \frac{1}{(V_0')^{1/3}} \quad \Rightarrow \quad y_{tt} - t\,y = 0$$

d) ~~For~~ Bounded solution as t gets large takes form for t large and negative,

$$y = \frac{A\Delta^{1/4}}{\cancel{}} \frac{\cos\left[\frac{2}{3}\sqrt{V_0'}\,(x_0-x)^{3/2} - \frac{\pi}{4}\right]}{(x_0-x)^{1/4}}$$

$$= \frac{A\Delta^{1/4}}{(x_0-x)^{1/4}} \cos\left[\frac{\pi}{4} - \frac{2}{3}\sqrt{V_0'}\,(x_0-x)^{3/2}\right]$$

Note: functional form matches that from (b)

Matching the phase of y above with the even solution in (b) yields

$$\frac{\pi}{4} = \varphi_0 - n\pi \quad \Rightarrow \quad \varphi_0 = n\pi + \frac{\pi}{4}$$

$$\Rightarrow \quad 2\varphi_0 = \underbrace{2n\pi}_{\text{even number}} + \frac{\pi}{2} \qquad (1)$$

Matching with the odd solution, gives

$$\frac{\pi}{4} = \varphi_0 - n\pi - \frac{\pi}{2}$$

$$\Rightarrow \varphi_0 = \frac{\pi}{4} + \left(n+\tfrac{1}{2}\right)\pi$$

$$2\varphi_0 = \underbrace{(2n+1)}_{\text{odd number}}\pi + \frac{\pi}{2} \qquad (2)$$

Putting together (1) and (2) yields

$$2\varphi_0 = \boxed{\int_{-x_0}^{x_0} dx\, k(x) = n\pi + \frac{\pi}{2} \qquad n = 0, 1, 2, \dots}$$

This is the B.S. quantization rule.

<u>Check on overlap</u>

How close to x_0 is the WKB solution in (6) valid?

$$k \simeq \sqrt{V_0'}\,(x_0 - x)^{1/2}$$

$$\frac{1}{L} \simeq \frac{1}{k^2}\frac{d}{dx}k^2 = \frac{1}{(x_0-x)} \qquad L \simeq x_0 - x$$

require $kL \gg 1 \Rightarrow \sqrt{V_0'}\,(x_0-x)^{3/2} \gg 1$

$\Rightarrow$ same as $t^{3/2} \gg 1$

Where is Airy Eqn valid? $\Rightarrow x_0 - x \ll x_0$

$\Rightarrow$ require $\sqrt{V_0'}\, x_0^{3/2} \gg 1$

What does this mean?

$$\underbrace{\sqrt{V_0' x_0}}\; x_0 \gg 1$$

This is an estimate for k a distance x_0 from the turning point x_0. We require

$$k x_0 \gg 1$$

Certainly valid for modes with high values of "n" in the quantization condition.
$\Rightarrow$ marginal for "n=0".

$\Rightarrow$ Actually the B.S.Q.R is surprisingly accurate even for n=0.