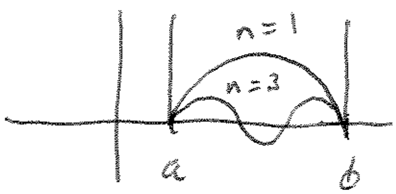
HWK # 12 Solutions

1. Solve $\dot{T} - \kappa T_{xx} = 0$ with $T = T_0$ at $x = a, b$

Take $T = 0$ for $x \in (a,b)$ at $t = 0$.

To satisfy BCs easiest to let $T = T_0 + \hat{T}$ with the BC. that $\hat{T} = 0$ at $x = a, b$.
The natural basis functions are zero at a, b

$$\Rightarrow \quad \sin \frac{n\pi(x-a)}{b-a} \equiv \varphi_n(x)$$



$\Rightarrow$ since initial conditions are symmetric around the center of the region only need odd values of n.

$$\hat{T}(x,t) = \sum_{n \text{ odd}} C_n(t)\, \varphi_n(x)$$ $\Rightarrow$ insert in eqn. above

~~Mult by $\varphi_m(x)$ and integrate in~~

$$\sum_n \left(\dot{C}_n + k_n^2 \kappa C_n\right) \varphi_n(x) = 0 \qquad k_n^2 = \frac{n^2\pi^2}{(b-a)^2}$$

$$\Rightarrow \dot{C}_n + k_n^2 \kappa C_n = 0$$

$$C_n(t) = C_n(0)\, e^{-k_n^2 \kappa t}$$

$$\hat{T}(x,t) = \sum_{n \text{ odd}} C_n(0)\, e^{-k_n^2 \kappa t}\, \varphi_n(x)$$

At $t = 0$, $T = 0 \Rightarrow \hat{T} = -T_0$

$$\sum_{n\,odd} C_n(0)\, \varphi_n(x) = -T_0$$

mult. by $\varphi_m(x)$ and integrate

$$C_m(0) \int_a^b dx\, \varphi_m^2 = -T_0 \int_a^b dx\, \sin k_m (x-a)$$

$$C_m(0)(b-a)\tfrac{1}{2} = +T_0 \left. \frac{\cos k_m(x-a)}{k_m} \right|_a^b$$

$$= \frac{T_0}{k_m}\left[\cos(m\pi) - 1\right]$$

$$= \frac{T_0}{k_m}\left((-1)^m - 1\right) = -\frac{2T_0}{k_m}$$

$$C_m(0) = -4\frac{T_0}{m\pi}$$

$$T = T_0 - T_0 \sum_{n\,odd} \frac{4}{n\pi} e^{-\frac{n^2\pi^2 K}{(b-a)^2} t} \sin\left[\frac{n\pi (x-a)}{b-a}\right]$$

Late time solution:

the larger values of n die quickly because of the exponential term.

To lowest order keep $n = 1$

$$T = T_0 \left(1 - \frac{4}{\pi} e^{-\frac{\pi^2 K}{(b-a)^2} t} \sin \frac{\pi (x-a)}{b-a}\right)$$

2 Solid sphere in 4-D world, ~~r~~ radius a.

$$\dot{T} - K \frac{1}{r^3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} T = 0$$

At $t=0$ $T = T_0$ $\Rightarrow$ plunge in heat bath with $T=0$

$\Rightarrow T(a) = 0$

a) How long for central temp. to change?

$\Rightarrow$ heat diffuses a distance "a"

$$\frac{\partial T}{\partial t} \sim \frac{T}{\tau} \sim K \frac{1}{a^2} T$$

$$\boxed{\tau \sim \frac{a^2}{K}}$$

b) Basis functions

Let $T(r,t) = \sum_n c_n(t)\, \mathcal{C}_n(r)$

Choose

$$\frac{1}{r^3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} \mathcal{C}_n(r) = -k_n^2 \mathcal{C}_n(r)$$

$$\mathcal{C}_n'' + \frac{3}{r}\mathcal{C}_n' + k_n^2 \mathcal{C} = 0$$

Write $\mathcal{C}_n = r^p \hat{\mathcal{C}}_n$ to try to convert to Bessel's Eqn

$$\cancel{r^{p}}\,\hat{a}_n'' + 2\,p\,r^{\cancel{p}-1}\hat{a}_n' + p(p-1)\,r^{p-2}\hat{a}$$

$$+ \frac{3}{r}\,\cancel{r^{p}}\,\hat{a}_n' + \frac{3}{r}\,p\,r^{\cancel{p}-1}\hat{a}_n + k_n^2\,\cancel{r^{p}}\,\hat{a}_n = 0$$

$$r^2\hat{a}_n'' + (2p+3)\,r\,\hat{a}_n' + \left[p(p-1) + 3p + k_n^2 r^2\right]\hat{a}_n = 0$$

To match Bessel's Eqn $2p + 3 = 1$

$$\Rightarrow p = -1 \quad \Rightarrow a_n = \frac{1}{r}\hat{a}_n$$

$$r^2\hat{a}_n'' + r\,\hat{a}_n' + \left[k_n^2 r^2 - 1\right]\hat{a}_n = 0$$

$\Rightarrow$ Bessel's eqn with $\nu = 1$

$$\hat{a}_n \sim J_1(k_n r),\ Y_1(k_n r)$$

Behavior near $r = 0$: $\Rightarrow$ Reg. Sing. Point

$$r^2 a_n'' + 3r\,a_n' + k_n^2 r^2 a_n = 0$$

near $r=0$

$$r^2 a_n'' + 3r\,a_n' = 0$$

$$p(p-1) + 3p = 0 \quad \Rightarrow p = 0,\ p = -2$$

$$a_n \sim r^0,\ r^{-2}$$

$$\Rightarrow \hat{a}_n \sim r,\ r^{-1}$$

$$\swarrow \qquad \searrow$$

$$J_1 \qquad\qquad Y_1$$

BCs ~~at r=0~~:

$$\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} \mathcal{R}_n(r) + k_n^2 r^3 \mathcal{R}_n(r) = 0$$

$$P(r) = r^3 , \quad w = r^3$$

$$r^3 \mathcal{R}_n(r) \mathcal{R}_m'(r) \Big|_0^a = 0$$

Take $\mathcal{R}_n(a) = 0$

$$r^3 \mathcal{R}_n(r) \mathcal{R}_m'(r) \Big|_{r=0} = 0$$

Solution with

$$\mathcal{R}_n = A_n \frac{1}{r} J_1(r) \sim r^0$$

$\Rightarrow$ ok

Solution with

$$\mathcal{R}_n \sim A_n \frac{1}{r} Y_1(k_n r) \sim \frac{1}{r^2}$$

$$r^3 \mathcal{R}_n \mathcal{R}_m' \Big|_{r=0} \neq 0$$

$\Rightarrow$ pick solution

$$\mathcal{R}_n = A_n \frac{1}{r} J_1(k_n r)$$

$$\mathcal{R}_n(a) = 0 = A_n \frac{1}{a} J_1(k_n a) = 0$$

$k_n a = X_{1n}$ = nth zero of the Bessel function of order 1.

$$\Rightarrow \quad \varphi_n = A_n \frac{1}{r} J_1\left(\frac{x_{1n} r}{a}\right)$$

$$n = 1, 2, \ldots\ldots$$

Orthogonality

$$\int_0^a dr\, r^3 \frac{J_1(k_n r)}{r} \frac{J_1(k_m r)}{r} = 0$$

for $n \neq m$

$\Rightarrow$ since φ_n satisfy self-adjoint equation and BCs.

normalization:

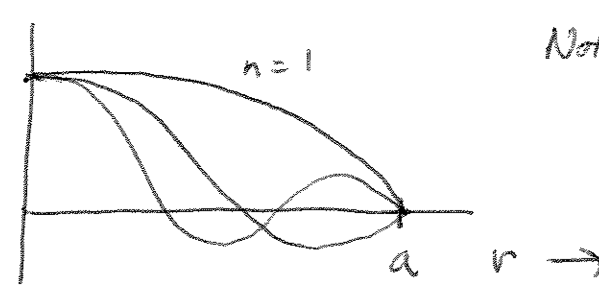
$$\int_0^a dr\, r^3 \varphi_n^2 = 1 = A_n^2 \int_0^a dr\, r\, J_1^2\left(\frac{x_{1n} r}{a}\right)$$

$$= A_n^2 a^2 \int_0^1 ds\, s\, J_1^2(x_{1n} s)$$

$$= A_n^2 a^2 \frac{1}{2} J_2^2(x_{1n})$$

$$A_n = \frac{1}{a} \sqrt{2} \frac{1}{J_2(x_{1n})}$$

$$\boxed{\varphi_n(r) = \frac{\sqrt{2}}{a} \frac{1}{J_2(x_{1n})} \frac{1}{r} J_1\left(\frac{x_{1n} r}{a}\right)}$$



Note: eigenfunctions are finite at $r = 0$ since $\varphi_n \sim r^0$

c) Insert series in diff. equ.

$$\sum_n \left(\dot{c}_n + \kappa k_n^2 c_n\right) \phi_n(r) = 0$$

$$\dot{c}_n + \kappa k_n^2 c_n = 0 \qquad c_n = c_n(0)\, e^{-k_n^2 \kappa t}$$

At $t=0$

$$T_0 = \sum_n c_n(0)\, \phi_n(r)$$

$$T_0 \int_0^a dr\, r^3 \phi_m(r) = c_m(0)$$

$$c_m(0) = A_m \int_0^a dr\, r^2\, J_1\!\left(x_{1n} \frac{r}{a}\right) T_0$$

$$= T_0\, a^3 A_m \int_0^1 ds\, s^2\, J_1(x_{1n} s)$$

$$\frac{d}{dx} x^n J_n(x) = x^n J_{n-1}(x)$$

$$\frac{d}{dx} x^2 J_2(x) = x^2 J_1(x)$$

$$\frac{1}{x_{1n}} \frac{d}{ds} s^2 J_2(x_{1n} s) = s^2 J_1(x_{1n} s)$$

$$c_m(0) = T_0 a^3 \frac{A_m}{x_{1m}} J_2(x_{1m}) = T_0 \frac{a^3}{x_{1m}} \frac{\sqrt{2}}{a} = \frac{\sqrt{2}\, a^2\, T_0}{x_{1m}}$$

$$\boxed{T(r,t) = T_0 \sum_n \frac{\sqrt{2}\, a^2}{x_{1n}}\, e^{-\frac{x_{1n}^2}{a^2}\kappa t}\, \phi_n(r)}$$

Late time $\quad T \sim \dfrac{\sqrt{2}\, a^2}{x_{11}} T_0\, e^{-\frac{x_{11}^2}{a^2}\kappa t}\, \phi_1(r) \Rightarrow$ exponential