Homework # 1 Solutions

6.1.10 (a)

$$\sin(x+iy) = \frac{1}{2i}\left[e^{ix} e^{i^2 y} - e^{-ix} e^{-i^2 y} \right]$$

$$= \frac{1}{2i}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right]$$

$$= \frac{1}{2}\left[\sin x\, e^{-y} + \sin x\, e^{y} + i(\cos x\, e^{y} - \cos x\, e^{-y})\right]$$

$$= \sin x \cosh y + i \cos x \sinh y$$

similar for $\cos(x+iy)$

6.1.15

a) Find the zeros of $\sin z$. From above

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$\Rightarrow$ must have both real and imag. part zero.

Since $\cosh y > 0$ everywhere,

$$\sin x = 0 \Rightarrow \boxed{x = n\pi}$$ n an integer

Since $\cos x = \cos n\pi \neq 0$, must have $\sinh y = 0$

$$\Rightarrow \boxed{y = 0}$$

d) Find zeros of $\cosh z$.

$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

Since $\cosh x > 0 \implies \cos y = 0 \implies \boxed{y = (n+\tfrac{1}{2})\pi}$

Since $\sin(n+\tfrac{1}{2})\pi \neq 0$, must have $\sinh x = 0$

$$\implies \boxed{x = 0}$$

6.2.1 a)

For $w = u + iv$ with w analytic show that $\nabla^2 u = \nabla^2 v = 0$

CR conditions: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= 0$$

$\nabla^2 v$ similar

6.2.2 Is $f(z) = \mathrm{Re}(z) = x$ analytic?

$$\implies v = 0$$

CR requires, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ with $u = x$

but $\dfrac{\partial u}{\partial x} \neq 0$ so not analytic anywhere.

6.2.3 For $w = u + iv$ with w analytic, show that u, v can not have max or min in region where ~~ ~~ w analytic.

Proof 1

Suppose u has max at z_0. Taylor series around this point

$$u(x,y) \simeq u(x_0,y_0) + \cancelto{0}{\frac{\partial u}{\partial x_0}}(x-x_0) + \cancelto{0}{\frac{\partial u}{\partial y_0}}(y-y_0) + \frac{1}{2}\frac{\partial^2 u}{\partial x_0^2}(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 u}{\partial y_0^2}(y-y_0)^2 + \frac{\partial^2 u}{\partial x_0 \partial y_0}(x-x_0)(y-y_0)$$

Along x direction:

$$u(x,y_0) = u(x_0,y_0) + \frac{1}{2}\frac{\partial^2 u}{\partial x_0^2}(x-x_0^2)$$

Along y direction:

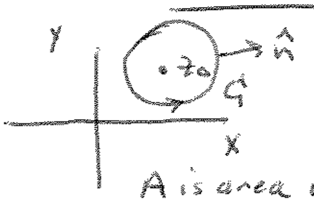
$$u(x_0,y) = u(x,y_0) + \frac{1}{2}\frac{\partial u}{\partial y_0^2}(y-y_0)^2$$

If u hax max at z_0 then $\frac{\partial^2 u}{\partial x_0^2} < 0$ but since $\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial y_0^2} > 0$ so along y direction u increases

$\Rightarrow$ no maximum

$\Rightarrow$ only a saddle point.

Proof 2 Use divergence theorem for real functions



$$\int_A dA\, \nabla^2 u = 0 = \int_A dA\, \nabla \cdot \nabla u = \int_C d\underset{\sim}{l}\, \hat{n} \cdot \nabla u$$

A is area within C $\Rightarrow$ $\hat{n} \cdot \nabla u$ has pos. and neg. values

$\Rightarrow$ z_0 can't be a max. or min.

6.2.8 Find CR conditions in polar coordinates.

$$f = R(r,\theta)\, e^{i\Theta(r,\theta)} \quad , \quad z = re^{i\theta}$$

$$df = dR\, e^{i\Theta} + R e^{i\Theta} i\, d\Theta$$

$$dz = dr\, e^{i\theta} + r e^{i\theta} i\, d\theta$$

$$\frac{df}{dz} = \frac{(dR + iR\, d\Theta)\, e^{i\Theta}}{(dr + ri\, d\theta)\, e^{i\theta}}$$

In r direction $\Rightarrow d\theta = 0$

$$\frac{df}{dz} = \left(\frac{\partial R}{\partial r} + iR\frac{\partial \Theta}{\partial r}\right)(\quad)$$

In θ direction $\Rightarrow dr = 0$

$$\frac{df}{dz} = \left(-i\frac{1}{r}\frac{\partial R}{\partial \theta} + \frac{R}{r}\frac{\partial \Theta}{\partial \theta}\right)(\quad)$$

$\Rightarrow$ must be same in both directions

$$\boxed{\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta} \quad , \quad \frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}}$$

6.7.1 a) Find how circles in z plane transform.

$$w = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$\Rightarrow$ eliminate θ $\Rightarrow$ $\sin^2\theta + \cos^2\theta = 1$

$$\frac{v^2}{\left(r - \frac{1}{r}\right)^2} + \frac{u^2}{\left(r + \frac{1}{r}\right)^2} = 1$$

In w plane have an ellipse with major and minor axis given by

$$\left|r - \frac{1}{r}\right|, \quad r + \frac{1}{r}$$

As $r \to 1$, becomes a line with $v = 0$ and $|u| \leq 2$.

6.7.3 a)

$$w(z) = \sin z = \sin x \cosh y + i \cos x \sinh y$$

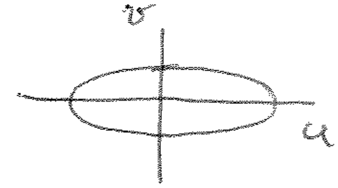
$$u = \sin x \cosh y, \quad v = \cos x \sinh y$$

For $y = c_2$ eliminate x,

$$\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1$$

$\Rightarrow$ ellipses with major, minor axes

$\cosh c_2$, $|\sinh c_2|$

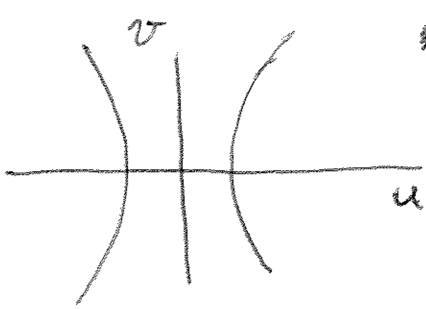


For $x = c_1$ eliminate y,

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1$$

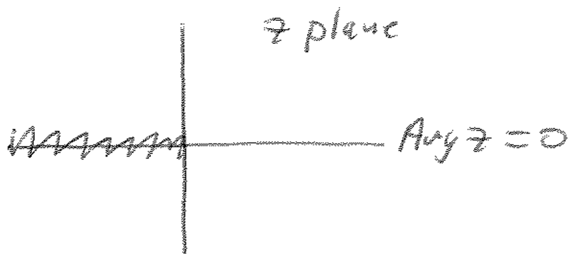
$\Rightarrow$ hyperbolas which intersect the u axis at $u = \pm \sin x$

* w plane



Note that the contours of const. x and const. y are $\perp$ to each other as they cross.

2 Define a cut in z plane to make $z^{1/3}$ single valued.



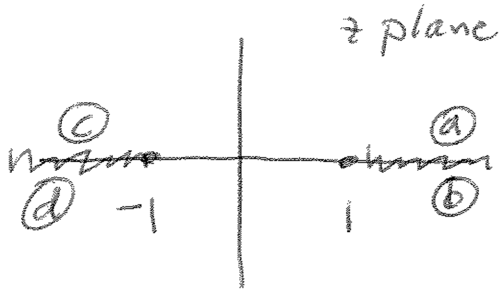
$\Rightarrow$ cut not unique

$$\mathrm{Arg}(-i)^{\frac{1}{3}} = \mathrm{Arg}\left(e^{-i\frac{\pi}{2}}\right)^{\frac{1}{3}}$$
$$= \mathrm{Arg}\left(e^{-i\frac{\pi}{6}}\right)$$
$$= -\frac{\pi}{6}$$

3 Define cuts to make

$$f = \ln(z^2-1)$$

single valued. Choose cut with symmetry



Can write

$$f = \ln(z-1) + \ln(z+1)$$

$\Rightarrow$ branch points at $z = \pm 1$

$\Rightarrow$ not unique. Can choose any cuts that end at $z = \pm 1$ and extend to ∞ in any direction.

Above cut with $x > 1$ $(y \gtrsim 0)$, define

$$\mathrm{Arg}(z-1) = 0, \quad \mathrm{Arg}(z+1) = 0$$

At (a), $\mathrm{Arg}(z-1) = 0$, $\mathrm{Arg}(z+1) = 0$

$$\Rightarrow \boxed{\mathrm{Im}\, f = 0}$$

At (b), $\mathrm{Arg}(z-1) = 2\pi$, $\mathrm{Arg}(z+1) = 0$

$$\mathrm{Im}\, f = \cancel{\mathrm{Im}(e^{2\pi i})} \; \mathrm{Im}(\ln e^{2\pi i})$$

$$= \boxed{2\pi}$$

At (c), $\mathrm{Arg}(z-1) = \pi$, $\mathrm{Arg}(z+1) = \pi$

$$\mathrm{Im} f = \mathrm{Im}(\ln(e^{\pi i})) + \mathrm{Im}(\ln e^{i\pi})$$

$$= \boxed{2\pi}$$

At (d), $\mathrm{Arg}(z-1) = \pi$, $\mathrm{Arg}(z+1) = -\pi$, $\boxed{\mathrm{Im} f = 0}$