Physics 604 Final Exam Fall ~~'04~~ '05

Dr. Drake

1. (a) (20 points) Solve for the lowest order behavior of the solution of the spherical Bessel equation near $z = 0$,

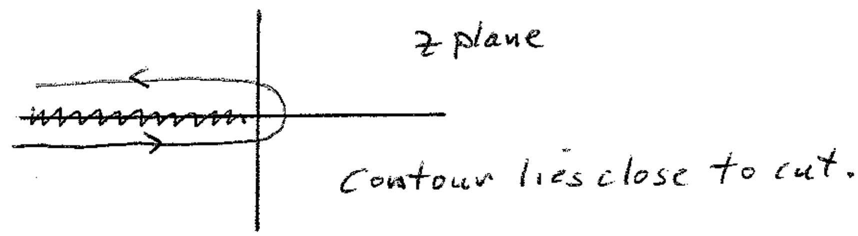
$$z^2 y'' + 2zy' + [z^2 - n(n+1)]y = 0,$$

and show that your two solutions are linearly independent.

(b) (30 points) Evaluate

$$\int_C \frac{dz}{z^\nu (z^2 + a^2)}$$

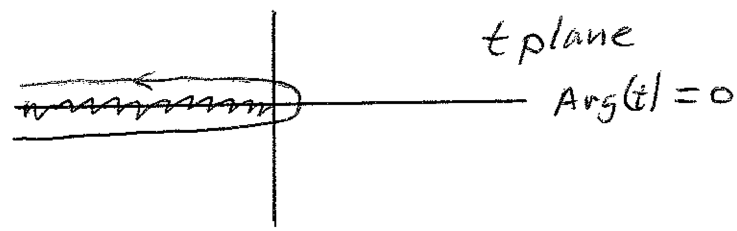
with a real, ν real and positive, and the contour C given by



2. (40 points) Consider the following integral

$$Q_\nu(x) = \int_C t^\nu e^{xt} dt$$

with ν real and positive and the contour C given by



(a) For what values of complex x is the integral defined?

(b) Evaluate $Arg(t^\nu)$ above and below the cut.

(c) For ν real and positive express $Q_\nu(x)$ in terms of the Gamma function,

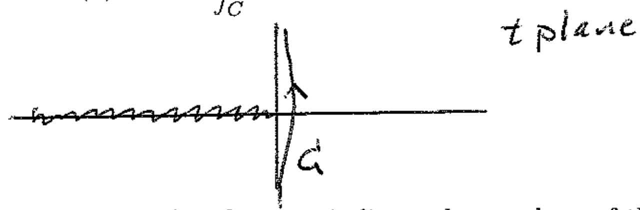
$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

(d) For what values of complex x is the result in (c) valid?

3. (40 points) An integral representation for the parabolic cylinder function is

$$D_\nu(x) = e^{x^2/4} \int_C t^\nu e^{-xt} e^{t^2/2} dt$$

with C given by



(a) For x large, real and positive and with $\nu \sim 1$ indicate the topology of the integrand in the t plane, locate the position of the saddle points and the PSD.

(b) Evaluate the integral for x large and positive.

4. (70 points) A liquid is heated in a hollow cylinder of radius b by a source at $r = r_0$ which is turned on at $t = 0$. The equation satisfied by the liquid is

$$\frac{\partial w}{\partial t} - \kappa \nabla^2 w = AH(t)\delta(r - r_0)$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

and the Heaviside function $H(t)$ is zero for negative argument and unity for positive argument. The temperature of the liquid is zero for $t < 0$ and remains zero at the boundary for all time.

(a) Construct a set of basis functions $\phi_n(r)$ to describe the liquid in the cavity. Write the basis functions as a linear combination of the two solutions of the Bessel equation $J_\nu(kr)$ and $Y_\nu(kr)$. Give expressions for the eigenvalues of your basis functions (in terms of the known properties of $J_\nu(kr)$ and $Y_\nu(kr)$) and define the normalization so that the $\phi_n(r)$ have unity norm. State why the basis functions are orthogonal (you don't have to prove orthogonality). What is the behavior of the eigenfunctions near $r = 0$? Sketch the lowest three eigenfunctions.
Hint: Bessel's equation is $r^2 y'' + ry' + (k^2 r^2 - \nu^2)y = 0$.

(b) Write $w(r, t)$ as

$$w(r, t) = \sum_{n=1}^{\infty} C_n(t)\phi_n(r)$$

and derive an equation for $C_n(t)$. What is the characteristic damping rate of the eigenfunctions?

(c) Solve for $C_n(t)$ and then write a solution for the complete space/time dependence of w.

(d) What is the form of the solution at late time? Does it reach a steady state?