Note that the Arfken problems correspond to the 7th Edition with the corresponding 6th Edition problems in the parentheses.

1. Arfken 20.2.4, 20.2.14, 20.3.6 (15.4.3). For 15.4.3 also calculate the solution by integrating across $x = 0$ to obtain the jump in the first derivative and then match to the solutions in the outer region.

2. The temperature in a one-dimensional medium satisfies a diffusion equation with diffusion coefficient D. At $t = 0$ a very localized source of heat is turned on. The equation satisfied by T is

$$\frac{\partial T}{\partial t} - D\frac{\partial^2}{\partial x^2}T = H(t)\delta(x)$$

where $H = 0$ for $t < 0$ and $H = 1$ for $t > 0$. Assume that $T = 0$ for $t < 0$.

   (a) Solve for the space/time dependence of $T(x,t)$ by first completing a Laplace transform and then a Fourier transform of the equation and then completing the subsequent inverse Laplace transform. The inversion of the Fourier transform can not be done exactly so you will have an integral representation for the solution given by a k space integral.

   (b) Consider the time dependence of T at $x = 0$. The k space integral can again not be carried out but by defining an appropriate dimensionless variable, you can extract the dependence of $T(0,t)$ on t and D. Does $T(0,t)$ reach a steady state or not? For what values of x is the $x = 0$ solution approximately valid?

   (c) Consider the time dependence of T for x large and positive. You should be able to evaluate the integral asymptotically in this limit. Be careful around $k = 0$. Is there a singularity? For a given value of x when does the asymptotic expansion of the integral break down?

3. Bohr-Sommerfeld Quantization Rule (optional problem)
   Consider an equation of the form

$$\frac{d^2y}{dx^2} - V(x)y = 0, \tag{1}$$

where $V(x)$ is symmetric around $x = 0$, has turning points at $x = \pm x_0$, $[V(\pm x_0) = 0]$ and $V(x) < 0$ for $|x| < x_0$. Thus, the local wavevector is given by $k(x) = [-V(x)]^{1/2}$. Assume that k is sufficiently large that the WKB solution to the equation is valid in regions away from the turning points. The goal of this problem is to find the condition under which bounded solutions to the equation exist.

(a) Write the even and odd WKB solutions to (1) valid in the region $|x| < x_0$. Why can you find even and odd solutions? Demonstrate that your solutions have this property.

(b) Evaluate the solution in (a) in the vicinity of the turning point at $x = x_0$. Assume that $(dV/dx)_{x=x_0} \neq 0$.
Hint: Use $\int_0^x dx() = \int_0^{x_0} dx() + \int_{x_0}^x dx()$ and evaluate the second integral for x close to x_0.

(c) Write an approximate form of (1) valid near the turning point and define a new independent variable t such that the equation takes the form of Airy's equation,
$$\frac{d^2 y}{dt^2} - ty = 0. \tag{2}$$

(d) The solution of (2) which is bounded as $t \to +\infty$ has the following form for t large and negative,
$$y = A\frac{1}{(-t)^{1/4}} \cos[\frac{2}{3}(-t)^{3/2} - \frac{\pi}{4}]$$
with A a constant. Write this solution in terms of the original dependent variable x.

(e) The functional form of the solutions in (b) and (d) should be identical. These solutions must match so that you must demand that the coefficient of the right (left) going wave in (b) equal that in (d). Do this for both even and odd solutions. These constraints yield the Bohr-Sommerfeld quantization rule.
$$\int_{-x_0}^{x_0} dx k(x) = (n + \frac{1}{2})\pi$$
where $n = 0, 1, 2, ...$