$\frac{x}{2}$.

ultiple-slit diffraction pattern. Another appli-
ιenon, Section 14.5.
l to form a geometric series (compare Sec-

ry–Perot interferometer.

and the hyperbolic functions are defined for
er series

$$/2\,\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s \frac{z^{2s+1}}{(2s+1)!},$$

$$\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s \frac{z^{2s}}{(2s)!},$$

$$\frac{z^{2s+1}}{(2s+1)!},$$

$$\sum_{=0}^{\infty} \frac{z^{2s}}{(2s)!}.$$

$z, \quad \sin iz = i \sinh z,$
$z, \quad \cos iz = \cosh z.$

ns such as

$h\, z = \frac{e^z + e^{-z}}{2},$

$\text{in}\, z_1 \cos z_2 + \sin z_2 \cos z_1,$

6.1.10 Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

established from comparison of power series, show that

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$

$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$

This demonstrates that we may have $|\sin z|, |\cos z| > 1$ in the complex plane.

6.1.11 From the identities in Exercises 6.1.9 and 6.1.10 show that

(a) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$

$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$

(b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y, \qquad |\cosh z|^2 = \cosh^2 x + \sin^2 y.$

6.1.12 Prove that

(a) $|\sin z| \ge |\sin x|$ (b) $|\cos z| \ge |\cos x|$.

6.1.13 Show that the exponential function e^z is periodic with a pure imaginary period of $2\pi i$.

6.1.14 Show that

(a) $\tanh \frac{z}{2} = \frac{\sinh x + i \sin y}{\cosh x + \cos y}$, (b) $\coth \frac{z}{2} = \frac{\sinh x - i \sin y}{\cosh x - \cos y}$.

6.1.15 Find all the zeros of

(a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

6.1.16 Show that

(a) $\sin^{-1} z = -i \ln\left(iz \pm \sqrt{1 - z^2}\right)$, (d) $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$,

(b) $\cos^{-1} z = -i \ln\left(z \pm \sqrt{z^2 - 1}\right)$, (e) $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

(c) $\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)$, (f) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$.

Hint. 1. Express the trigonometric and hyperbolic functions in terms of exponentials. 2. Solve for the exponential and then for the exponent.

6.1.17 In the quantum theory of the photoionization we encounter the identity

$$\left(\frac{ia - 1}{ia + 1}\right)^{ib} = \exp\left(-2b \cot^{-1} a\right),$$

Example 6.2.2 z^* IS NOT ANALYTIC

Let $f(z) = z^*$. Now $u = x$ and $v = -y$. Applying the Cauchy–Riemann conditions, we obtain

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy–Riemann conditions are not satisfied and $f(z) = z^*$ is not an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane.

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighborhood—for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far-reaching implications. The real and imaginary parts of our analytic function must separately satisfy Laplace's equation. This is Exercise 6.2.1. Further, our analytic function is guaranteed derivatives of all orders, Section 6.4. In this sense the derivative not only governs the local behavior of the complex function, but controls the distant behavior as well. ■

Exercises

6.2.1 The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.

(a) Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's equation such as $u(x, y)$ and $v(x, y)$ are called **harmonic** functions.

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0,$$

and give a geometric interpretation.

Hint. The technique of Section 1.6 allows you to construct vectors normal to the curves $u(x, y) = c_i$ and $v(x, y) = c_j$.

6.2.2 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

6.2.3 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$ and $v(x, y)$ **cannot both have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have saddle points only.)

6.2.4 Let $A = \partial^2 w/\partial x^2$, $B = \partial^2 w/\partial$
two variables, $w(x, y)$, we have

With $f(z) = u(x, y) + iv(x, y$
neither $u(x, y)$ nor $v(x, y)$ **ha**
complex plane. (See also Sectic

6.2.5 Find the analytic function

v

if (a) $u(x, y) = x^3 - 3xy^2$, (b)

6.2.6 If there is some common regi
$u(x, y) - iv(x, y)$ are both anal

6.2.7 The function $f(z) = u(x, y) +$

6.2.8 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Phi(r,\theta}$
functions of r and θ, show tha
become

(a) $\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}$, (b) $\frac{1}{r}\frac{\partial R}{\partial \theta}$

Hint. Set up the derivative first

6.2.9 As an extension of Exercise 6.2.
coordinates. Equation (2.35) (w
polar coordinates.

6.2.10 Two-dimensional irrotational fl
tial $f(z) = u(x, v) + iv(x, y)$.
and the imaginary part, $v(x, y)$
$\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

(a) Show that $df/dz = V_x - i$
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no s
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrc

6.2.11 A proof of the Schwarz inequal

$f = \psi_{aa}$

The ψ are integrals of products
λ is a complex parameter.

(a) Differentiate the preceding
dent parameter, independe
to zero yields

plying the Cauchy–Riemann conditions, we

$$\frac{\partial v}{\partial y} = -1.$$

fied and $f(z) = z^*$ is not an analytic function continuous, thus providing an example of a where differentiable in the complex plane. ariable is essentially a local characteristic, in only in a local neighborhood — for instance, e of a derivative of a function of a complex ions. The real and imaginary parts of our an- e's equation. This is Exercise 6.2.1. Further, s of all orders, Section 6.4. In this sense the r of the complex function, but controls the ■

real and imaginary parts, respectively, of an

s exist, show that

$$= \nabla^2 v = 0.$$

as $u(x, y)$ and $v(x, y)$ are called **harmonic**

$$+ \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0,$$

ou to construct vectors normal to the curves

$(z) = x$ is analytic.

d the imaginary part $v(x, y)$ of an analytic tion, show that $u(x, y)$ and $v(x, y)$ **cannot** **num** in the interior of any region in which

6.2.4 Let $A = \partial^2 w/\partial x^2$, $B = \partial^2 w/\partial x \partial y$, $C = \partial^2 w/\partial y^2$. From the calculus of functions of two variables, $w(x, y)$, we have a **saddle point** if

$$B^2 - AC > 0.$$

With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy–Riemann conditions and show that **neither** $u(x, y)$ nor $v(x, y)$ **has a maximum or a minimum** in a finite region of the complex plane. (See also Section 7.3.)

6.2.5 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

if (a) $u(x, y) = x^3 - 3xy^2$, (b) $v(x, y) = e^{-y}\sin x$.

6.2.6 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

6.2.7 The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.

6.2.8 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Phi(r,\theta)}$, in which $R(r, \theta)$ and $\Phi(r, \theta)$ are differentiable real functions of r and θ, show that the Cauchy–Riemann conditions in polar coordinates become

(a) $\dfrac{\partial R}{\partial r} = \dfrac{R}{r}\dfrac{\partial \Theta}{\partial \theta}$, (b) $\dfrac{1}{r}\dfrac{\partial R}{\partial \theta} = -R\dfrac{\partial \Theta}{\partial r}$.

Hint. Set up the derivative first with δz radial and then with δz tangential.

6.2.9 As an extension of Exercise 6.2.8 show that $\Theta(r, \theta)$ satisfies Laplace's equation in polar coordinates. Equation (2.35) (without the final term and set to zero) is the Laplacian in polar coordinates.

6.2.10 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

(a) Show that $df/dz = V_x - iV_y$;
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks);
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

6.2.11 A proof of the Schwarz inequality (Section 10.4) involves minimizing an expression,

$$f = \psi_{aa} + \lambda\psi_{ab} + \lambda^*\psi_{ab}^* + \lambda\lambda^*\psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex and λ is a complex parameter.

(a) Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f/\partial\lambda^*$ equal to zero yields

$$\lambda = -\frac{\psi_{ab}^*}{}.$$

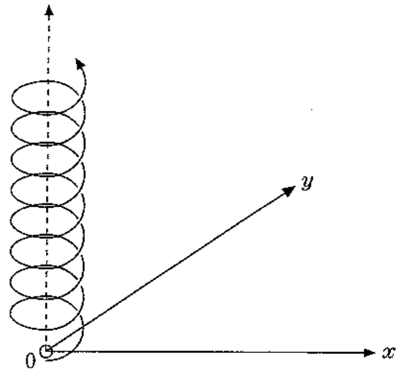

FIGURE 6.23 This is the Riemann surface for $\ln z$, a multivalued function.

line $\theta = 0$ (positive real axis) as a cut line. This is equivalent to taking one and only one complete turn of the spiral staircase.

The concept of mapping is a very broad and useful one in mathematics. Our mapping from a complex z-plane to a complex w-plane is a simple generalization of one definition of function: a mapping of x (from one set) into y in a second set. A more sophisticated form of mapping appears in Section 1.15 where we use the Dirac delta function $\delta(x - a)$ to map a function $f(x)$ into its value at the point a. Then in Chapter 15 integral transforms are used to map one function $f(x)$ in x-space into a second (related) function $F(t)$ in t-space.

Exercises

6.7.1 How do circles centered on the origin in the z-plane transform for

(a) $w_1(z) = z + \dfrac{1}{z}$, (b) $w_2(z) = z - \dfrac{1}{z}$, for $z \neq 0$?

What happens when $|z| \to 1$?

6.7.2 What part of the z-plane corresponds to the interior of the unit circle in the w-plane if

(a) $w = \dfrac{z-1}{z+1}$, (b) $w = \dfrac{z-i}{z+i}$?

6.7.3 Discuss the transformations

(a) $w(z) = \sin z$, (c) $w(z) = \sinh z$,

(b) $w(z) = \cos z$, (d) $w(z) = \cosh z$.

Show how the lines $x = c_1$, $y = c_2$ map into the w-plane. Note that the last three transformations can be obtained from the first one by appropriate translation and/or rotation.

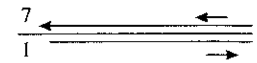

FIGURE 6.24

6.7.4 Show that the function

is single-valued if we take −1

6.7.5 Show that negative numbers h
ln(−1).

6.7.6 An integral representation of
shown in Fig. 6.24. Map this
examples of mapping are giver

6.7.7 For noninteger m, show that th
suitably defined branch of the
why $|z| < 1$ may be taken as th
in light of the cut you have cho

6.7.8 The Taylor expansion of Exer
than the one suitably defined b
that other branches cannot hav
guishable.] Using the same br
find the corresponding Taylor
coefficients.

6.8 CONFORMAL MAPPIN

In Section 6.7 hyperbolas were m
into circles. Yet in all these trans
was a result of the fact that all the
As long as $w = f(z)$ is an anal