Physics 604 Midterm #2 Fall '20

Dr. Drake

1. (35 points) The Coulomb Wave equation is given by

$$\rho \frac{d^2 w}{d\rho^2} + (\rho - 2\eta) w = 0 \tag{1}$$

where η is real and $\rho > 0$. This equation describes a wave propagating towards a turning point and an adjacent singularity.

   (a) Classify the singular points of this equation (don't worry about infinity).

   (b) Find the behavior of the solutions for ρ very close to zero. Show that the solutions are linearly independent.

   (c) Plot the square of the WKB wavevector $k^2(\rho)$ versus ρ. Calculate the behavior of the solutions for ρ very large.

2. (30 points) An integral representation for the Coulomb wave equation in (1) can be written as

$$w(\rho) = \int_C dk Y(k) e^{ik\rho}. \tag{2}$$

   (a) Derive a differential equation for Y(k) by inserting (2) into (1). Explicitly give the conditions on the contour C in order that the integral solution satisfy (1). The solution of your equation for $Y(k)$ is given by (you don't need to show this)

$$Y(k) = \frac{1}{1-k^2} \left(\frac{1+k}{1-k} \right)^{-i\eta}. \tag{3}$$

   (b) Choose two contours in the complex k plane which satisfy the constraint given in part (a) and therefore give two linearly independent solutions to (1). Draw them.

   Hint: You must define cuts for the singularities of $Y(k)$ at $k = \pm 1$. The contours wrap around the cuts. Where do they end? Recall that $\rho > 0$.

3. (35 points) A wave in a one-dimensional medium is driven by a δ-function source at $x = 0$ that is turned on at $t = 0$ as follows:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} y = H(t)\delta(x) \tag{4}$$

where $y = 0$ for $t < 0$ and $H(t)$ is zero for $t < 0$ and one for $t > 0$.

   (a) Solve for the space/time dependence of $y(x,t)$ by completing Laplace and Fourier transforms of the equation and then completing the subsequent inverse Laplace transform. The k space integral that you will obtain is given by

$$y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{k^2 c^2} \left(1 - \cos(kct)\right) \tag{5}$$

(b) Finally complete the inverse Fourier transform to obtain $y(x,t)$. Plot the spatial dependence of y at a fixed time. Interpret the result on the basis of causality. Hint: To reduce the required algebra evaluate $y(x,t)$ explicitly only $x > 0$ and use the symmetry of the problem to sketch the solution for $x < 0$.

Midterm #2 Solutions

① $$\rho w_{\rho\rho} + (\rho - 2\eta) w = 0$$

a)

Singular point at $\rho = 0$, regular elsewhere

$$\rho^2 w_{\rho\rho} + \underbrace{(\rho^2 - 2\eta\rho)}_{\text{analytic at } \rho = 0} w = 0$$

$\Rightarrow$ reg. sing. pt at $\rho = 0$

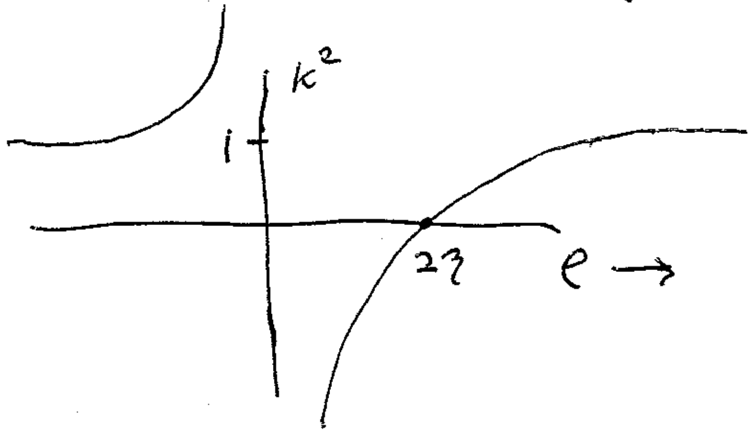
b) Near $\rho = 0$

$$w_{\rho\rho} = 0 \quad \Rightarrow \quad w_1 = a, \quad w_2 = b\rho$$

$$W = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$$

$\Rightarrow$ linearly indep.

c) $$k^2 = 1 - \frac{2\eta}{\rho} \qquad w_{\rho\rho} + \left(1 - \frac{2\eta}{\rho}\right) w = 0$$



$$k \simeq 1 - \frac{\eta}{\rho}$$ for large ρ

large ρ $\Rightarrow$ WKB solution

$$w \sim \frac{e^{\pm i \int^{\rho} d\rho' \left(1 - \frac{\eta}{\rho'}\right)}}{1}$$

$$= e^{\pm i\rho \mp i\eta \ln(\rho)}$$

$$= \frac{e^{\pm i\rho}}{e^{\pm i\eta}}$$

3 (6)

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} y = H(t)\,\delta(x)$$

Laplace transform and Fourier trans.

$$\int_{-\infty}^{\infty} dx \int_0^{\infty} dt\, e^{i\omega t} e^{-ikx} \left[\qquad \right]$$

$\Rightarrow$ no contributions from end points since $y=0$ at $x=\pm\infty$ and $y=0$ at $t=0$

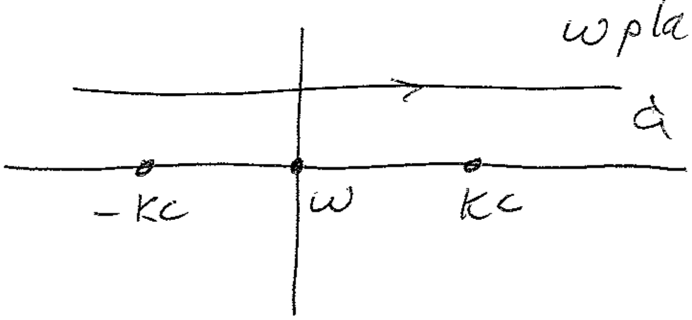
$$-\omega^2 Y(k,\omega) + k^2c^2\, Y(k,\omega) = \int_{-\infty}^{\infty} dx\, \delta(x) \int_0^{\infty} dt\, e^{i\omega t}$$

$$= \left.\frac{e^{i\omega t}}{i\omega}\right|_0^{\infty} = -\frac{1}{i\omega}$$

and $\operatorname{Im}\omega > 0$ so endpoint at $t=\infty$ is zero.

$$Y(k,\omega) = \frac{1}{i\omega}\,\frac{1}{\omega^2 - k^2c^2}$$

$$Y(k,t) = \frac{1}{i}\int_{C} \frac{d\omega}{2\pi} e^{i\omega t}\, \frac{1}{\omega}\, \frac{1}{(\omega-kc)(\omega+kc)}$$



⟹ three singularities

For $t>0$ close contour in LHP. By Jordan's Lemma contribution from semi-circle is zero. Calculate residue from three poles.

$$\tilde{Y}(k,t) = \frac{1}{i}\left(\frac{-2\pi i}{2\pi}\right)\left[\frac{1}{-k^2c^2} + \frac{e^{ikct}}{2k^2c^2} + \frac{e^{-ikct}}{2k^2c^2}\right]$$

$$= \frac{1}{k^2c^2}\left[1 - \cos(kct)\right]$$

$$y(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}\, \frac{1}{k^2c^2}\left(1-\cos kct\right)$$

⟹ note that no singularity at $k=0$ since $\cos kct \sim 1 - \frac{k^2c^2t^2}{2}$ for k small

⟹ move contour above $k=0$

$$y = \frac{1}{2\pi}\int_{C'} dk\, \cancel{e^{ikx}}\, \frac{1}{k^2c^2}\left[e^{ikx} - \frac{1}{2}e^{ik(x-ct)} - \frac{1}{2}e^{ik(x+ct)}\right]$$

$$= I_1 + I_2 + I_3$$

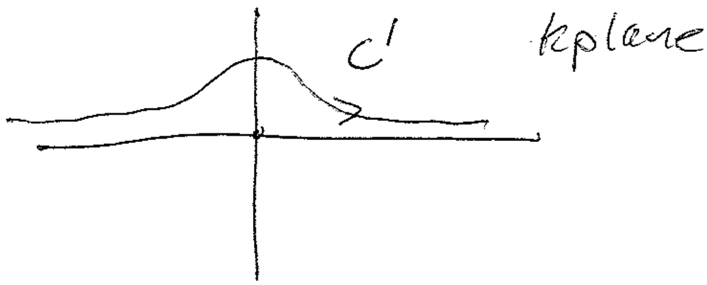
consider only $x>0$

⟹ y is even function of x

$$I_1 = \frac{1}{2\pi c^2}\int_{c'} dk \, \frac{1}{k^2} e^{ikx}$$

$\Rightarrow$ close in UHP

$\Rightarrow$ no enclosed poles



$$I_1 = 0$$

$$I_2 = -\frac{1}{2\pi}\frac{1}{2}\frac{1}{c^2}\int_{c'} dk \, \frac{1}{k^2} e^{ik(x-ct)}$$

for $x > ct$ close in UHP

$\Rightarrow 0$

for $x < ct$ close in LHP

$$I_2 = -\frac{-2\pi i}{2\pi}\frac{1}{2c^2}\frac{d}{dk} e^{ik(x-ct)}\Big|_{k=0}$$

$$= \frac{i}{2c^2} i(x-ct) = \frac{ct-x}{2c^2}$$

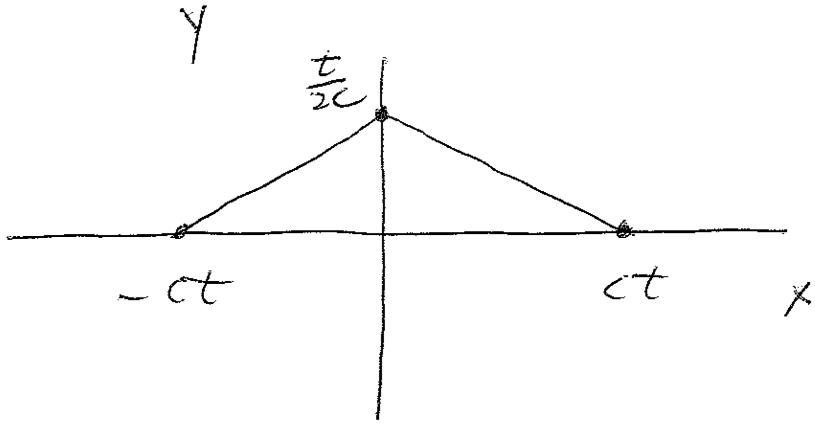
$$I_2 = 0 \qquad x > ct$$

$$= \frac{ct-x}{2c^2} \qquad x < ct$$

$$I_3 = -\frac{1}{2\pi}\frac{1}{2c^2}\int_{c'} dk \, \frac{1}{k^2} e^{ik(x+ct)}$$

for all $x > 0$ close in UHP

$$\Rightarrow I_3 = 0$$



Note that $y = 0$ for $x > ct$

$\Rightarrow$ wave propagates at velocity c so no wave beyond $|x| = ct$.

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a)

$$e \frac{d^2}{de^2} w + (e - 2q) w = 0$$

Let $w = \int_C dk \, e^{ike} Y(k)$

$$0 = \int_C dk \left[(1-k^2) e - 2q \right] Y e^{ike}$$

$$= \int_C dk \, Y \left[(1-k^2)\left(-i \frac{d}{dk}\right) - 2q \right] e^{ike}$$

$\Rightarrow$ integrate by parts

$$(1-k^2) Y e^{ike} \Big|_{\text{end points of } C} = 0$$

$$0 = \int_C dk \, e^{ike} \left[i \frac{d}{dk} (1-k^2) Y - 2q Y \right]$$

$$\Rightarrow \quad i \frac{d}{dk} (1-k^2) Y - 2q Y = 0$$

define $\hat{Y} = (1-k^2) Y$

$$i \hat{Y}' - 2q \hat{Y} / (1-k^2) = 0$$

$$\int \frac{d\hat{Y}}{\hat{Y}} = -\tfrac{1}{2} i\eta \int dk \left(\frac{1}{1-k} + \frac{1}{1+k}\right)\frac{1}{k}$$

$$\ln \hat{Y} = -i\eta \ln\left(\frac{1+k}{1-k}\right)$$

$$\hat{Y} = \left(\frac{1+k}{1-k}\right)^{-i\eta}$$

$$y = \frac{1}{1-k^2}\left(\frac{1+k}{1-k}\right)^{-i\eta} = \frac{(1+k)^{-i\eta-1}}{(1-k)^{-i\eta+1}}$$

e) constraint

$$\hat{Y} e^{ike}\Big| = 0$$

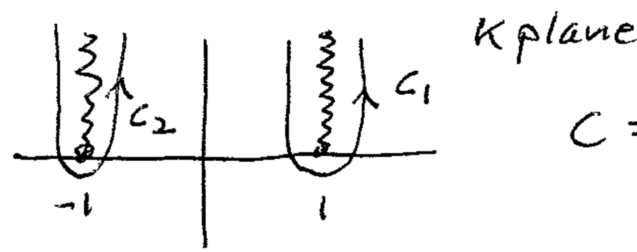
$$\left[\frac{(1+k)}{(1-k)}\right]^{-i\eta} e^{ike} = 0$$

$$(1 \pm k)^{\mp i\eta} \neq 0 \text{ at } k = \pm 1$$

$$\Rightarrow e^{ike}\Big| = 0$$

$$\Rightarrow k \to i\infty$$

Choose cuts from $k = \pm 1$



$C = C_1$ or C_2