Physics 604 Exam #3 Fall '10

Dr. Drake

1. (30 points) A spherical object of radius a is immersed in a zero temperature heat bath. The sphere is heated with a δ-function source at $r = r_0$. The steady state diffusion equation for the temperature of the sphere is

$$-\kappa \nabla^2 T = A\delta(r - r_0),$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

Solve for the radial dependence of the temperature once the temperature of the sphere has reached a steady state. Sketch the temperature versus radius.

2. (30 points) Consider a conducting, two-dimensional rectangular cavity of width a in the x direction with a bottom at $y = 0$ and infinite extent in the positive y direction. The electric potential V is maintained at zero on the two sides and V_0 on the bottom. The potential $V(x, y)$ satisfies Poisson's equation

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V = 0$$

(a) Write the solution for V in terms of a separable set of eigenfunctions $X_m(x)$ and $Y_m(y)$ as follows:

$$V(x, y) = \sum_m c_m X_m(x) Y_m(y)$$

Write equations for $X_m(x)$ and $Y_m(y)$ and solve these equations using boundary conditions appropriate for the solution V. Normalize the basis functions so they have unit norm.

(b) Solve for the c_m by matching the form of the potential at $y = 0$.

(c) Write the lowest order, non-trivial, behavior of the solution for large y. What is the asymptotic value of V at large y? What is the characteristic scale length over which V approaches this value?

3. (40 points) Consider an azimuthally symmetric wave on a cylindrical drum with a radius b. The displacement y of the surface of the drum satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \nabla^2 y = 0,$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

and $y(r, t = 0) = y_0 \delta(r - r_0)$ with $\dot{y}(r, t = 0) = 0$.

(a) Construct a set of basis functions $\phi_n(r)$ to solve for $y(r, t)$. Write the basis functions as a linear combination of the two solutions of the Bessel equation $J_\nu(kr)$ and $Y_\nu(kr)$. Give expressions for the eigenvalues of your basis functions (in terms of the known properties of $J_\nu(kr)$ and $Y_\nu(kr)$) and define the normalization so that the $\phi_n(r)$ have unity norm. State why the basis functions are orthogonal (you don't have to prove orthogonality). What is the behavior of the eigenfunctions near $r = 0$? Sketch the lowest three eigenfunctions.
Hint: Bessel's equation is $r^2 f'' + r f' + (k^2 r^2 - \nu^2) f = 0$ and $\int_0^1 dr r [J_\nu(x_{\nu n} r)]^2 = [J_{\nu+1}(x_{\nu n})]^2/2$.

(b) Write $y(r, t)$ as

$$y(r, t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(r)$$

and derive an equation for $c_n(t)$. What is the characteristic frequency of the eigenfunctions?

(c) Solve for $c_n(t)$ and then write a solution for the complete space/time dependence of y.