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Current density and forces for a current loop moving parallel over a thin conducting sheet

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Abstract

An analytical expression for the eddy current forces on a circular current loop moving with constant velocity over a thin conducting sheet is derived in this paper. This calculation is based on using the boundary conditions across the conducting sheet to solve for the total magnetic vector potential. The eddy currents in the sheet and the forces on the sheet are first presented in the 'quasi-static' limit and the perfect conductor limit. Finally, an analytical expression for both the drag and lift force is derived.

1. Introduction

There is a popular lecture demonstration in undergraduate physics courses that involves a metallic sheet swinging in a magnetic field. In the frame of reference of the sheet the magnetic field is changing; this changing magnetic field induces currents (called eddy currents) in the sheet. A Lorentz force acts on the eddy currents, causing a drag force (called magnetic drag) on the conducting sheet. This drag force can be enhanced by increasing the conductivity of the metal or by increasing the magnetic field. For example, placing the sheet in liquid nitrogen and repeating the demonstration after it has equilibrated to 77 K dramatically increases the eddy current damping force due to the increase in conductivity of the metallic sheet.

In this paper the induced currents and forces are analytically calculated for a circular dc current loop moving with a constant non-relativistic velocity in a parallel plane over a thin conducting sheet as shown in figure 1. To simplify the calculations in this paper we will assume that the thickness of the conducting sheet approaches the infinitesimal limit such that the 2D conductance of the sheet remains finite; i.e. $\sigma_\square = \lim_{t\to 0}(\sigma t) > 0$ where σ represents the bulk conductivity of the metallic film and t the thickness of the film. While eddy currents have theoretically been understood since the late 1800s (Maxwell 1872), a complete solution of the problem solved in this paper has not been found by the author. Our interest in

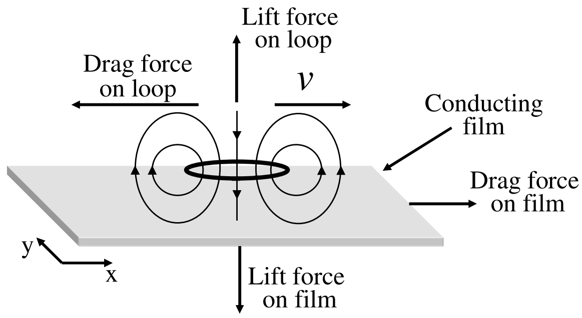

Figure 1. Pictorial setup for eddy current calculation. A circular current loop with a dc current I flowing in the clockwise direction is moving with a constant velocity $v\hat{x}$ over a thin conducting sheet.

understanding the currents and forces on the sheet is a result of our recent measurements of the eddy current drag force using high-Q mechanical oscillators with a spatial resolution of 100 μm (Palmer *et al* 2000). A more typical application of the analysis of eddy current forces is for magnetic levitation systems (MAGLEV).

The magnitude of the eddy current drag force can be estimated by using some simple arguments. For example, if the conducting sheet is stationary and the current loop is moving with a velocity $\mathbf{v} = v\hat{x}$, as shown in figure 1, using a relativistic argument there will be an electric field in the sheet's frame of reference given by $\mathbf{E}(\mathbf{r}) = -(\mathbf{v}/c)\times\mathbf{B}_{\text{loop}}(\mathbf{r})$ where $\mathbf{B}_{\text{loop}}(\mathbf{r})$ is the magnetic field from the current loop at point $\mathbf{r}$.[1] The electric field in the sheet gives rise to a current density $\mathbf{j}(\mathbf{r}) = \sigma_\square\mathbf{E}(\mathbf{r})$ which using the above estimate for the electric field has a y component given by $j_{\text{y}}(\mathbf{r}) = -\sigma_\square(v/c)(\mathbf{B}_{\text{loop}}(\mathbf{r})\cdot\hat{z})$. The magnetic field, from the current loop, exerts a Lorentz force on the current density in the sheet. The differential force at a point $\mathbf{r}$ on the sheet is given by $\mathrm{d}\mathbf{F}(\mathbf{r}) = \mathbf{j}(\mathbf{r})\times\mathbf{B}_{\text{loop}}(\mathbf{r})\,\mathrm{d}s$. The longitudinal force, or 'drag' force, is $F_x = \int j_{\text{y}}(\mathbf{B}_{\text{loop}}\cdot\hat{z})\,\mathrm{d}s$ and the 'lift' force is given by $F_z = \int[j_{\text{x}}(\mathbf{B}_{\text{loop}}\cdot\hat{y}) - j_{\text{y}}(\mathbf{B}_{\text{loop}}\cdot\hat{x})]\,\mathrm{d}s$ where the integral is a surface integral over the entire film. For example, the drag force on the sheet using the current density from the estimation above is

$$\mathrm{F}_x(\mathbf{r}) = \frac{\sigma_\square v}{c}\int(\mathbf{B}_{\text{loop}}(\mathbf{r})\cdot\hat{z})^2\,\mathrm{d}s. \tag{1}$$

Provided that the velocity of the current loop is such that $v \ll \mathrm{c}^2/(2\pi\sigma_\square)$, equation (1) is a good estimate of the force, as will be shown in the next section. This limit is just the 'quasi-static' limit for which the effects of screening currents can be neglected.

The estimate that produced equation (1) neglected conservation of current density and the screening response of the conductor. In the next section the total magnetic vector potential is solved. The outline of this calculation is as follows. The static magnetic vector potential of a circular current loop is the starting point to calculate the currents in the conducting sheet and the forces on the sheet. A Galilean transformation will be performed on the static magnetic vector potential to calculate the magnetic vector potential of the current loop in the sheet's frame of reference. Boundary conditions across the sheet will be used to calculate the magnetic vector potential associated with the currents formed in the sheet due to the changing magnetic field. After solving for the total magnetic vector potential, the current density in the sheet and the forces on the sheet will be investigated in two limits. Finally, in sections 3 and 4 an analytical expression for the drag force and lift force is analytically solved.

[1] In this paper, Gaussian units are used. See appendix A in Jackson (1975) for the conversion to SI.

2. Calculation of $\mathbf{A}_{\text{sheet}}$

The magnetic vector potential for a current loop of radius R, clockwise current I and with the centre of the loop located at a height $z' = b$ and at the origin in the (x', y') plane is given by the real part of

$$\mathbf{A}_{\text{loop}}(\mathbf{r}', t) = \hat{x}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha x'}\sin(\beta y')\exp[-k|z'-b|]\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$+\,\hat{y}\frac{IR}{c}\mathrm{i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha x'}\cos(\beta y')\exp[-k|z'-b|]\,\mathrm{d}\alpha\,\mathrm{d}\beta \quad (2)$$

where $k = \sqrt{\alpha^2+\beta^2}$ and J_1 is the Bessel function of the first order (see appendix A for a derivation of $\mathbf{A}_{\text{loop}}$). In the frame of reference of the conducting sheet, a Galilean transformation on equation (2) can be performed for $v \ll c$ by replacing $x' = x - vt$ ($y' = y$ and $z' = z$) in equation (2).

The response of the conducting sheet will be calculated using a method originally devised by Maxwell (1872) and more recently applied by Reitz (1970). In this method, the continuous motion of the loop is broken up into an infinite number of instantaneous infinitesimal steps. The infinite number of steps are added up and a limit is taken to get the continuous motion of the loop.

Before calculating the infinite series, first consider one instantaneous step of the centre of the current loop from $x = -v\delta t$ to $x = v\delta t$ at $t = 0$. For $t < 0$, x in equation (2) is replaced by $x \to x + v\delta t$ and for $t > 0$, $x \to x - v\delta t$. At $t = 0$, currents are produced in the sheet to maintain the initial magnetic field and hence the initial magnetic vector potential (Smythe 1950). Therefore the magnetic vector potential associated with the response of the sheet, which we denote as $\mathbf{A}_{\text{sheet}}$, in the plane of the sheet ($z = 0$) and at $t = 0$, is the magnetic vector potential of the loop at $x = v\delta t$ minus at $x = -v\delta t$

$$\mathbf{A}_{\text{sheet}}(x, y) = \hat{x}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}\,\mathrm{e}^{\mathrm{i}\alpha x}J_1[kR](\mathrm{e}^{\mathrm{i}\alpha v\delta t}-\mathrm{e}^{-\mathrm{i}\alpha v\delta t})\sin(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$+\,\hat{y}\mathrm{i}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}\,\mathrm{e}^{\mathrm{i}\alpha x}J_1[kR](\mathrm{e}^{\mathrm{i}\alpha v\delta t}-\mathrm{e}^{-\mathrm{i}\alpha v\delta t})\cos(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \quad (3)$$

Boundary conditions determine how the currents in the sheet evolve in time. From the discontinuity of the $\mathbf{H}$ field across the sheet, the following boundary condition for the magnetic vector potential can be derived (Smythe 1950)

$$\left.\frac{\partial\mathbf{A}_{\text{sheet}}}{\partial z}\right|_{z=-\epsilon}^{\epsilon} = \frac{4\pi\mu\sigma_\Box}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}_{\text{sheet}}+\mathbf{A}_{\text{loop}})\Big|_{z=0}. \quad (4)$$

For the case of a single step at $t = 0$ the magnetic vector potential associated with the loop is not changing for $t \neq 0$. In this case, for $t > 0$, equation (4) reduces to

$$\left.\frac{\partial\mathbf{A}_{\text{sheet}}}{\partial z}\right|_{z=-\epsilon}^{\epsilon} = \frac{4\pi\mu\sigma_\Box}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_{\text{sheet}}. \quad (5)$$

The currents associated with $\mathbf{A}_{\text{sheet}}$ are confined within the sheet or the $z = 0$ plane so that $\mathbf{A}_{\text{sheet}} \propto \mathrm{e}^{-k|z|}$, therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_{\text{sheet}} = -kw\mathbf{A}_{\text{sheet}} \quad (6)$$

where $w = c^2/2\pi\mu\sigma_\Box$ has dimensions of velocity[2]. Solving this differential equation yields $\mathbf{A}_{\text{sheet}} = \mathbf{A}_{\text{sheet}}(z = 0, t = 0)\,\mathrm{e}^{-k|z|}\,\mathrm{e}^{-kwt}$ where $\mathbf{A}_{\text{sheet}}(z = 0, t = 0)$ is given by equation (3).

[2] In MKS units, $w = 2/\mu_\circ\sigma_\Box$. We follow the same notation that Reitz (1970) used in defining this parameter as w. Saslow (1992) used the notation $v_\circ$ for this parameter. For the rest of this paper we set $\mu = 1$.

This implies that an instantaneous change in the magnetic field produces currents in the sheet that decay with a decay rate inversely proportional to $\sigma_\Box$. As the conductivity increases, the currents take a longer time to decay. Another way to interpret this result is that a change in the current loop produces an image on the opposite side of the sheet that recedes away from the $z = 0$ plane. This is how Maxwell (1872) envisioned eddy currents as discussed in Saslow's article (1992).

The exact solution of $\mathbf{A}_{\rm sheet}$ can be derived by using the initial conditions at $t = 0$. For $t > 0$ the currents in the sheet decay so that

$$\mathbf{A}_{\rm sheet}(\mathbf{r}) = \hat{x}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}\,\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{e}^{-kwt}J_1[kR]2\mathrm{i}\sin(\alpha v\delta t)\sin(\beta y)\,\mathrm{e}^{-k(|z|+b)}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$+\hat{y}\mathrm{i}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}\,\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{e}^{-kwt}J_1[kR]2\mathrm{i}\sin(\alpha v\delta t)\cos(\beta y)\,\mathrm{e}^{-k(|z|+b)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{7}$$

Equation (7) considers only one discrete jump in the exciting magnetic field. To calculate the eddy current distribution when the loop has moved continuously from $x = -\infty$ to the origin ($x = 0$), the sum of discrete steps from $t = -\infty$ to $t = 0$ is calculated followed by the limit of an infinite number of infinitesimal steps. Taking the terms from equation (7) that depend on x and t the nth step would be

$$\mathrm{e}^{\mathrm{i}\alpha x}\left[\mathrm{e}^{\mathrm{i}\alpha v(n+1/2)\delta t} - \mathrm{e}^{\mathrm{i}\alpha v(n-1/2)\delta t}\right]\mathrm{e}^{-kw(n\delta t)} = \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{e}^{\mathrm{i}\alpha v(n\delta t)}2\,\mathrm{i}\sin(\alpha v\delta t/2)\,\mathrm{e}^{-kw(n\delta t)}. \tag{8}$$

Adding up all of the steps and taking the limit $\delta t \to 0$ yields

$$\lim_{\delta t\to 0}\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}\alpha v(n\delta t)}\sin(\alpha v\delta t/2)\,\mathrm{e}^{-kw(n\delta t)} = \frac{\alpha v}{2}\int_0^{\infty}\mathrm{e}^{\mathrm{i}\alpha vt}\,\mathrm{e}^{-kwt}\,\mathrm{d}t$$
$$= \frac{\alpha v}{2}\frac{kw + i\alpha v}{(\alpha v)^2 + (kw)^2}. \tag{9}$$

Therefore $\mathbf{A}_{\rm sheet}$ at $t = 0$ when the loop has moved continuously from $x = -\infty$ to $x = 0$ is

$$\mathbf{A}_{\rm sheet}(\mathbf{r}) = \hat{x}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha x}\sin(\beta y)\,\mathrm{e}^{-k(|z|+b)}\mathrm{i}\alpha v\frac{kw + \mathrm{i}\alpha v}{(\alpha v)^2 + (kw)^2}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$+\hat{y}\mathrm{i}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha x}\cos(\beta y)\,\mathrm{e}^{-k(|z|+b)}\mathrm{i}\alpha v\frac{kw + \mathrm{i}\alpha v}{(\alpha v)^2 + (kw)^2}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{10}$$

For $t \neq 0$, x in equation (10) is replaced by $x \to x - vt$. The total magnetic vector potential is $\mathbf{A}_{\rm total} = \mathbf{A}_{\rm loop} + \mathbf{A}_{\rm sheet}$ where $\mathbf{A}_{\rm loop}$ is given by equation (2) in the frame of reference of the conducting sheet and $\mathbf{A}_{\rm sheet}$ is given by equation (10). Since the total magnetic vector potential is, in principle, solved, the current density in the sheet and the total magnetic field can be calculated.

2.1. Quasi-static limit

The first limit that shall be investigated is when the conductivity is small such that $v \ll w$, which is the quasi-static limit. In this limit, $kw \gg \alpha v$ and equation (10) becomes

$$\mathbf{A}_{\rm sheet}(\mathbf{r}, t) = \hat{x}i\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\sin(\beta y)\,\mathrm{e}^{-k(|z|+b)}\frac{\alpha v}{kw}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$-\hat{y}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\cos(\beta y)\,\mathrm{e}^{-k(|z|+b)}\frac{\alpha v}{kw}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{11}$$

$\mathbf{A}_{\text{sheet}}$ is v/w smaller than $\mathbf{A}_{\text{loop}}$, therefore $\mathbf{A}_{\text{total}} \simeq \mathbf{A}_{\text{loop}}$. The shielding currents from the conducting sheet in this limit decay instantly since the conductivity of the sheet is small, and hence contribute negligibly to the currents in the sheet. The total current density in the sheet is given by $\mathbf{j} \simeq -\frac{\sigma_\square}{c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_{\text{loop}}\big|_{z=0}$ and at $t = 0$ is

$$\mathbf{j}(x, y) \overset{t=0}{=} -\hat{x}\frac{IR\sigma_\square v}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha\beta}{k^2}J_1[kR]\sin(\alpha x)\sin(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$-\hat{y}\frac{IR\sigma_\square v}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha^2}{k^2}J_1[kR]\cos(\alpha x)\cos(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \quad (12)$$

The dimensionality of this integral can be reduced by switching the α and β integrals to polar components. Making the substitution $\alpha = k\cos(\theta)$ and $\beta = k\sin(\theta)$,

$$\mathbf{j}(x, y) \overset{t=0}{=} -\hat{x}\frac{IR\sigma_\square v}{c^2}\int_0^{\infty}kJ_1[kR]\,\mathrm{e}^{-kb}\int_0^{2\pi}\cos(\theta)\sin(\theta)\sin[xk\cos(\theta)]\sin[yk\sin(\theta)]\,\mathrm{d}\theta\,\mathrm{d}k$$
$$-\hat{y}\frac{IR\sigma_\square v}{c^2}\int_0^{\infty}kJ_1[kR]\,\mathrm{e}^{-kb}\int_0^{2\pi}\cos^2(\theta)\cos[xk\cos(\theta)]\cos[yk\sin(\theta)]\,\mathrm{d}\theta\,\mathrm{d}k. \quad (13)$$

Most of the azimuthal integrals in this paper have this form. To assist the reader, the integrals of this nature relevant to this paper have been tabulated in appendix B. Using equations (B.4) and (B.3), equation (13) becomes

$$\mathbf{j}(x, y) \overset{t=0}{=} -\hat{x}\frac{2\pi IR\sigma_\square v}{c^2}\frac{xy}{\rho^2}\int_0^{\infty}kJ_1[kR]J_2[k\rho]\,\mathrm{e}^{-kb}\,\mathrm{d}k$$
$$-\hat{y}\frac{\pi IR\sigma_\square v}{c^2}\int_0^{\infty}kJ_1[kR]\left(J_0[k\rho]+\frac{y^2-x^2}{\rho^2}J_2[k\rho]\right)\mathrm{e}^{-kb}\,\mathrm{d}k. \quad (14)$$

Figure 2 is a vector plot of the current density (equation (14)) in the sheet in the limit $v \ll w$ for $b = R/2$.

The last things to calculate in this limit are the forces on the sheet. The three components of the magnetic field from the loop in the plane of the sheet ($z = 0$) and at $t = 0$ are

$$\mathbf{B}_{\text{loop}}\cdot\hat{x} \overset{t=0}{=} \frac{2\pi IR}{c}\int_0^{\infty}\frac{x}{\rho}kJ_1[kR]J_1[k\rho]\,\mathrm{e}^{-kb}\,\mathrm{d}k \quad (15)$$

$$\mathbf{B}_{\text{loop}}\cdot\hat{y} \overset{t=0}{=} \frac{2\pi IR}{c}\int_0^{\infty}\frac{y}{\rho}kJ_1[kR]J_1[k\rho]\,\mathrm{e}^{-kb}\,\mathrm{d}k \quad (16)$$

$$\mathbf{B}_{\text{loop}}\cdot\hat{z} \overset{t=0}{=} -\frac{2\pi IR}{c}\int_0^{\infty}kJ_1[kR]J_0[k\rho]\,\mathrm{e}^{-kb}\,\mathrm{d}k. \quad (17)$$

The drag force on the sheet in the quasi-static limit using equations (17) and (14) can now be calculated. The $\hat{y}$ component of the current density has two components; the first term has spatial polar symmetry whereas the second term depends on x and y. When calculating the total force by integrating over the entire sheet, this second term goes to zero since the z component of the magnetic field has spatial polar symmetry. Investigation of the remaining integral shows that the total longitudinal force on the sheet is

$$F_x = \frac{\sigma_\square v}{2c}\int_0^{\infty}2\pi(\mathbf{B}_{\text{loop}}(\rho)\cdot\hat{z})^2\rho\,\mathrm{d}\rho = \frac{v}{2w}\frac{(2\pi IR)^2}{c}\int_0^{\infty}kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k \quad (18)$$

where the closure relation for the Bessel function $\left(\int_0^{\infty}\rho J_0[k'\rho]J_0[k\rho]\,\mathrm{d}\rho = \frac{1}{k}\delta(k-k')\right)$ has been used in the second equality. This force is half of the force that was originally estimated

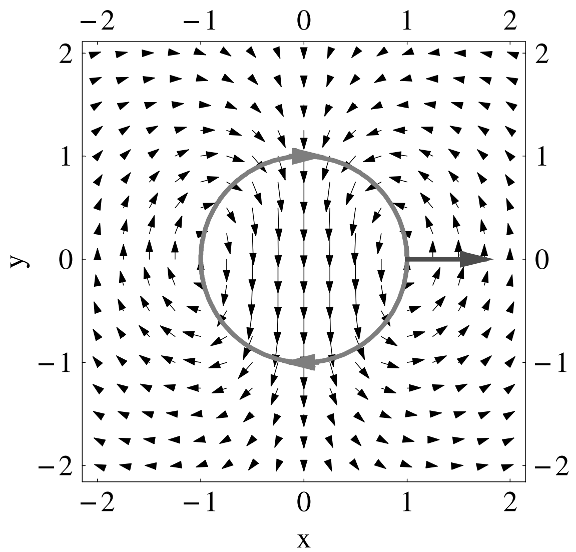

Figure 2. Eddy current density in the limit $v \ll w$ (quasi-static limit). The circle denotes the location of the current loop with a dc clockwise current flowing in the loop. The arrow denotes the direction of the velocity of the loop. The height of the loop was $b = R/2$ for this calculation.

in section 1 (equation (1))[3]. The lift force due to the symmetry of both the current density (equation (14)) and the magnetic field (equations (15) and (16)) is zero in this limit.

The question arises: why does the force that has been calculated differ from the back of the envelope calculation by a factor of two in the limit when the velocity is small? Investigation of this factor of two difference in the force initially started out in the calculation of the electric field, i.e. the electric field (current density) used in equation (1) is a factor of two larger than the electric field (current density) used in equation (18). When the magnetic vector potential of the current loop was transformed to the moving coordinate system, this transformation neglected the scalar potential because that term does not play a role in the current density. Let us investigate this term further.

The magnetic vector potential forms a 4-vector with the electric scalar potential. When the magnetic vector potential was transformed to a moving coordinate system, the correct transformation should have yielded an electric scalar potential as well as the magnetic vector potential. For $v \ll c$, the correct transformation of the magnetic vector potential to the moving coordinate system yields the following scalar potential:

$$\varphi(\mathbf{r}) = \frac{4IRv}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\cos[\alpha(x-vt)]\sin[\beta y]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{19}$$

The electric field from the scalar potential is given by

$$\begin{aligned}\mathbf{E}_\varphi(\mathbf{r}) = {}& \hat{x}\frac{4IRv}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha\beta}{k^2}J_1[kR]\sin[\alpha(x-vt)]\sin(\beta y)\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &-\hat{y}\frac{4IRv}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta^2}{k^2}J_1[kR]\cos[\alpha(x-vt)]\cos(\beta y)\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &-\hat{z}\frac{4IRv}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta k}{k^2}J_1[kR]\cos[\alpha(x-vt)]\sin(\beta y)\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \end{aligned}\tag{20}$$

[3] The currents and the force were calculated in a different manner by Salzman *et al* (2001).

This electric field is similar in form and magnitude to the electric field from the magnetic vector potential of the loop. There are two differences between this electric field and the form in equation (12). The first difference is that the x-component in $\mathbf{E}_\varphi$ has the opposite sign from equation (12) so that instead of the electric field pointing outward, away from $x = 0$ as shown in the vector plot of the current density in figure 2, this electric field points inward (towards $x = 0$). The electric field also points perpendicular to the sheet (i.e. $\mathbf{E}_\varphi$ has a z-component) which is the second difference.

The electric field perpendicular to the sheet produces surface charges on the sheet and hence an electrostatic field that cancels this electric field. Because of this cancellation, the scalar potential does not produce any current that contributes to the eddy current forces. Thus the drag force in the quasi-static limit is $1/2$ of what is expected from the back of the envelope calculation (equation (1)).

2.2. Perfect conductor

The other limit that shall be investigated is the perfect conductor limit. In this limit $w \to 0$ (i.e. the currents do not decay), $v \gg w$, and the second part of the integrand in equation (10) can be approximated as the following:

$$\mathrm{i}\alpha v \frac{kw + \mathrm{i}\alpha v}{(\alpha v)^2 + (kw)^2} \simeq -\frac{(\alpha v)^2\left(1 - \frac{\mathrm{i}kw}{\alpha v}\right)}{(\alpha v)^2} \simeq -\left[1 - \mathrm{i}\frac{kw}{\alpha v}\right]. \tag{21}$$

Substituting this result into equation (10)

$$\begin{aligned}\mathbf{A}_{\text{sheet}}(\mathbf{r}, t) = &-\hat{x}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\sin(\beta y)\,\mathrm{e}^{-k(|z|+b)}\left[1 - \mathrm{i}\frac{kw}{\alpha v}\right]\mathrm{d}\alpha\,\mathrm{d}\beta \\ &-\hat{y}\mathrm{i}\frac{IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\cos(\beta y)\,\mathrm{e}^{-k(|z|+b)}\left[1 - \mathrm{i}\frac{kw}{\alpha v}\right]\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned} \tag{22}$$

Comparing equation (22) to the vector potential for the moving current loop (equation (2)) it is noted that the first term in the square brackets cancels with the vector potential associated with the moving current loop so that the total vector potential in the plane of the sheet is

$$\begin{aligned}\mathbf{A}_{\text{total}}(x, y) \cong &\ \hat{x}\mathrm{i}\frac{IRw}{cv}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{\alpha k}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\sin(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &-\hat{y}\frac{IRw}{cv}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}J_1[kR]\,\mathrm{e}^{\mathrm{i}\alpha(x-vt)}\cos(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned} \tag{23}$$

The current density in the sheet at $t = 0$ is

$$\begin{aligned}\mathbf{j}(x, y) \overset{t=0}{=} &-\hat{x}\frac{IR}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k}J_1[kR]\cos(\alpha x)\sin(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &+\hat{y}\frac{IR}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k}J_1[kR]\sin(\alpha x)\cos(\beta y)\,\mathrm{e}^{-kb}\,\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned} \tag{24}$$

Switching the α and β integrals to polar coordinates and performing the azimuthal integral yields

$$\mathbf{j}(x, y) \overset{t=0}{=} -\frac{IRy}{\rho}\hat{x}\int_0^{\infty}kJ_1[k\rho]J_1[kR]\,\mathrm{e}^{-kb}\,\mathrm{d}k + \frac{IRx}{\rho}\hat{y}\int_0^{\infty}kJ_1[k\rho]J_1[kR]\,\mathrm{e}^{-kb}\,\mathrm{d}k. \tag{25}$$

Figure 3 is a vector plot of the current density in the limit that the conductance approaches infinity for $b = R/2$. In this limit the current density in the sheet acts as an image loop which shields the conducting sheet from changes in the magnetic field.

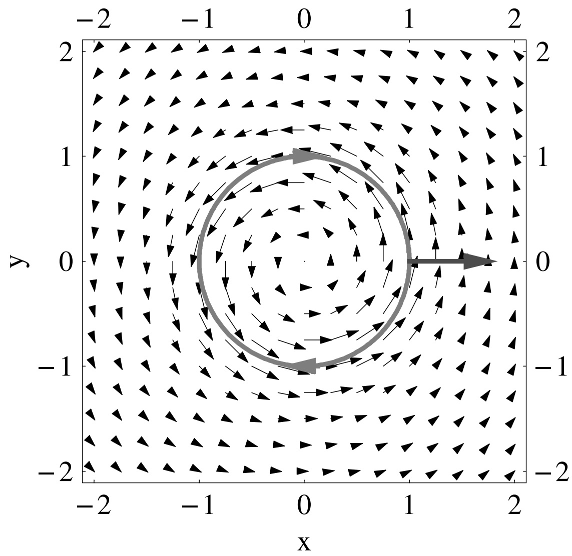

Figure 3. Eddy current density in the limit that $v \gg w$ (perfect conductor). The circle denotes the location of the current loop with a dc clockwise current flowing in the loop. The arrow denotes the direction of the velocity of the loop and the height of the loop was $b = R/2$ for this calculation.

To demonstrate that the magnetic field does not penetrate the conducting sheet in this limit, the z component of the magnetic field between the sheet and the loop will be calculated next. The total magnetic vector potential between the sheet and the loop $0 < z < b$ is given by

$$\mathbf{A}_{\text{total}}(\mathbf{r}) \stackrel{t=0}{\equiv} \frac{2IR}{c}\hat{x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta}{k^2}J_1[kR]\cos(\alpha x)\sin(\beta y)\,\mathrm{e}^{-kb}\sinh(kz)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$-\frac{2IR}{c}\hat{y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha}{k^2}J_1[kR]\sin(\alpha x)\cos(\beta y)\,\mathrm{e}^{-kb}\sinh(kz)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{26}$$

The z component of the total magnetic field is given by

$$\mathbf{B}_{\text{total}}(\mathbf{r})\cdot\hat{z} = -\frac{2IR}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J_1[kR]\cos(\alpha x)\cos(\beta y)\,\mathrm{e}^{-kb}\sinh(kz)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$= -\frac{2IR}{c}\int_0^{\infty}kJ_1[kR]\,\mathrm{e}^{-kb}\sinh(kz)\int_0^{2\pi}\cos[kx\cos(\theta)]\cos[ky\sin(\theta)]\,\mathrm{d}\theta\,\mathrm{d}k$$
$$= -\frac{4\pi IR}{c}\int_0^{\infty}kJ_1[kR]J_0[k\rho]\,\mathrm{e}^{-kb}\sinh(kz)\,\mathrm{d}k. \tag{27}$$

As $z \to 0$, the z component of the magnetic field goes to zero. This is the expected result for a perfect conductor.

Since the induced currents in the sheet have the same symmetry as the current loop the drag force is zero in this limit. The lift force given by $F_z = \int[j_x(\mathbf{B}_{\text{loop}}\cdot\hat{y}) - j_y(\mathbf{B}_{\text{loop}}\cdot\hat{x})]\,\mathrm{d}s$, on the other hand, is not zero. Using equations (25), (15) and (16) and switching the spatial coordinates in the integral over the entire sheet to cylindrical coordinates allows us to perform both the azimuthal integral (yielding a 2π) and radial integral (using the closure relation) as shown here:

$$F_z \overset{\sigma_\square\to\infty}{=} -2\pi\frac{(IR)^2}{c}\int_0^\infty k' J_1[k'R]\,\mathrm{e}^{-k'b}\int_0^\infty kJ_1[kR]\,\mathrm{e}^{-kb}\int_0^\infty \rho J_1[k'\rho]J_1[k\rho] \times\int_0^{2\pi}\mathrm{d}\phi\,\mathrm{d}\rho\,\mathrm{d}k\,\mathrm{d}k' = -\frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k. \tag{28}$$

3. Drag force

In this section an analytical expression for the drag force ($F_x = \int j_y(\mathbf{B}_{\mathrm{loop}}\cdot\hat{z})\,\mathrm{d}s$) is derived. The drag force can be written as $F_x = F_x^{(QS)} + F_x^{(R)}$ where $F_x^{(QS)}$ is the quasi-static drag force given by equation (18) and $F_x^{(R)}$ is the drag force due to the response of the conducting sheet to the changing magnetic field (i.e. the current density due to $\mathbf{A}_{\mathrm{sheet}}$: equation (10)), which was negligibly small in the quasi-static limit. To begin the calculation of $F_x^{(R)}$, the azimuthal integral can be calculated using equation (B.1) after switching to spatial cylindrical coordinates ($x = \rho\cos(\phi)$ and $y = \rho\sin(\phi)$).

Switching the α and β integrals to polar coordinates allows us to integrate the azimuthal dependence:

$$\int_0^{2\pi}\frac{\alpha^4}{\alpha^2 + k^2\frac{w^2}{v^2}}\,\mathrm{d}\theta = \int_0^{2\pi}k^2\frac{\cos^4(\theta)}{\cos^2(\theta)+\frac{w^2}{v^2}}\,\mathrm{d}\theta = \pi k^2\left(1 - 2\frac{w^2}{v^2} + \frac{2(w/v)^3}{\sqrt{1+(w/v)^2}}\right). \tag{29}$$

Using the closure relation for the Bessel function yields the following result for the drag force due to the response of the conductor:

$$F_x^{(R)} = -\frac{v}{2w}\left(1 - 2\left(\frac{w}{v}\right)^2 + \frac{2(w/v)^3}{\sqrt{1+(w/v)^2}}\right)\frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k. \tag{30}$$

The total drag force on the conducting sheet using equation (18) for the drag force in the quasi-static limit and equation (30) is

$$F_x = \frac{w}{v}\left(1 - \frac{w}{\sqrt{v^2+w^2}}\right)\frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k. \tag{31}$$

4. Lift force

Finally, an analytical expression for the lift force ($F_z = \int[j_{\mathrm{x}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{y}) - j_{\mathrm{y}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{x})]\,\mathrm{d}s$) on the sheet is derived. To calculate j_{x} and j_{y} we take the time derivatives of equations (2) and (10) and use equations (15) and (16) for the x and y components of the magnetic field. We start this calculation by switching the spatial coordinates in the integral over all of space to cylindrical coordinates, which allows us to integrate the azimuthal (ϕ) integral using equation (B.2).

Switching the α and β integrals to polar coordinates allows us to calculate the azimuthal (θ) integral:

$$\int j_{\mathrm{x}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{y})\,\mathrm{d}s \propto \int_0^{2\pi}\frac{\cos^2(\theta)\sin^2(\theta)}{\cos^2(\theta)+(w/v)^2}\,\mathrm{d}\theta = \pi\left(1 + 2\left(\frac{w}{v}\right)^2 - 2\frac{w}{v}\sqrt{1+\left(\frac{w}{v}\right)^2}\right) \tag{32}$$

and

$$\int j_{\mathrm{y}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{x})\,\mathrm{d}s \propto \int_0^{2\pi}\frac{\cos^4(\theta)}{\cos^2(\theta)+(w/v)^2}\,\mathrm{d}\theta = \pi\left(1 - 2\left(\frac{w}{v}\right)^2 + 2\frac{(w/v)^3}{\sqrt{1+\left(\frac{w}{v}\right)^2}}\right). \tag{33}$$

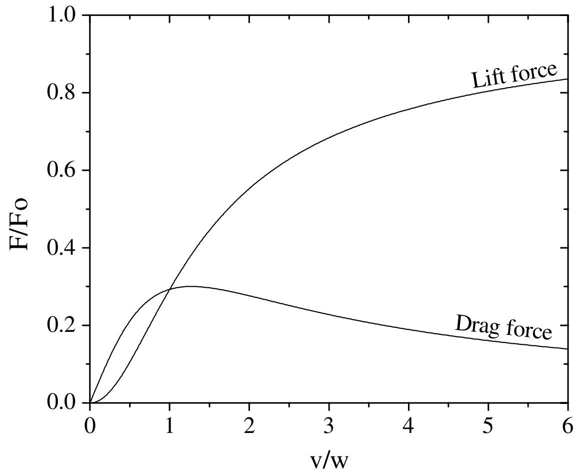

Figure 4. Drag (equation (31)) and lift force (equation (36)) as a function of v/w. The force (y axis) has been normalized to $F_\circ = \frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k$.

Using the closure relation for the Bessel function yields

$$\int j_{\mathrm{x}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{y})\mathrm{d}s = -\frac{2(\pi IR)^2}{c} \times \int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k\left(1+2\left(\frac{w}{v}\right)^2 - 2\frac{w}{v}\sqrt{1+\left(\frac{w}{v}\right)^2}\right) \quad (34)$$

and

$$\int j_{\mathrm{y}}(\mathbf{B}_{\mathrm{loop}}\cdot\hat{x})\,\mathrm{d}s = \frac{2(\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k\left(1-2\left(\frac{w}{v}\right)^2 + 2\frac{(w/v)^3}{\sqrt{1+\left(\frac{w}{v}\right)^2}}\right). \quad (35)$$

Subtracting equation (35) from equation (34), the total lift force on the sheet is

$$F_z = -\left(1-\frac{w}{\sqrt{v^2+w^2}}\right)\frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k. \quad (36)$$

Figure 4 shows the magnitude of both the drag and lift force on the sheet as a function of v/w normalized to $F_\circ = \frac{(2\pi IR)^2}{c}\int_0^\infty kJ_1^2[kR]\,\mathrm{e}^{-2kb}\,\mathrm{d}k$.

Aside from the normalization factor ($F_\circ$), equations (31) and (36) have the same functional dependence on v/w as the lift and drag force calculated for a moving magnetic monopole over a thin conducting sheet (Maxwell 1954, Reitz 1970 and Saslow 1992). This agreement is reasonable since the fields from the loop appear to be dipole in nature when far from the sheet and a magnetic dipole could be considered to consist of two magnetic monopoles.

5. Conclusions

An analytical expression for the forces on a current loop moving with constant velocity in the non-relativistic limit ($v \ll c$) over a conducting sheet has been derived in this paper. The ratio of the drag force (equation (31)) to the lift force (equation (36)) is $F_{\mathrm{drag}} = (w/v)F_{\mathrm{lift}}$. This result agrees with Davis's result (1972) in which it was derived using the Poynting vector and is a general result which does not depend on field geometry.

Under most experimental conditions, the quasi-static limit is the valid limit. For example, assume the velocity of the loop is $v = 25$ cm s^{-1} and the conducting sheet has a resistivity equal to 1 $\mu\Omega$ cm, which is approximately the resistivity of copper at room temperature. For a thickness of 1 mm, $\sigma_{\square} \simeq 9 \times 10^{16}$ cm s^{-1} making $w \simeq 1600$ cm s^{-1}. Using these numbers the deviation from the quasi-static limit is on the order of 0.01% for the drag force.

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Appendix A. Magnetic vector potential

To calculate the magnetic vector potential for the current loop in Cartesian coordinates (equation (2)) we start with the magnetic vector potential in cylindrical coordinates provided by Jackson (1975, problem 5.4):

$$\mathbf{A}_{\text{loop}}(\rho, z) = -\hat{\varphi}\frac{2\pi IR}{c}\int_0^\infty J_1[kR]J_1[k\rho]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}k. \tag{A.1}$$

Transforming equation (A.1) to Cartesian coordinates yields

$$\begin{aligned}\mathbf{A}_{\text{loop}}(\mathbf{r}) = \hat{x}\frac{2\pi IRy}{c\rho}&\int_0^\infty J_1[kR]J_1[k\rho]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}k\\ &-\hat{y}\frac{2\pi IRx}{c\rho}\int_0^\infty J_1[kR]J_1[k\rho]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}k.\end{aligned} \tag{A.2}$$

The Bessel function with the spatial dependence (e.g. $(y/\rho)J_1[k\rho]$ in the $\hat{x}$ term) can be converted into an integral representation using equation (B.2),

$$\begin{aligned}\mathbf{A}_{\text{loop}}(\mathbf{r}) = \hat{x}\frac{IR}{c}&\int_0^\infty\int_0^{2\pi}\frac{\sin(\theta)}{k^2}J_1[kR]\cos[kx\cos(\theta)]\sin[ky\sin(\theta)]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\theta\,\mathrm{d}k\\ &-\hat{y}\frac{IR}{c}\int_0^\infty\int_0^{2\pi}\frac{\cos(\theta)}{k^2}J_1[kR]\sin[kx\sin(\theta)]\cos[ky\cos(\theta)]\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\theta\,\mathrm{d}k.\end{aligned} \tag{A.3}$$

Finally we can convert the integrals into their Cartesian counterparts using $\alpha = k\cos(\theta)$, $\beta = k\sin(\theta)$, and $k^2 = \alpha^2 + \beta^2$:

$$\begin{aligned}\mathbf{A}_{\text{loop}}(\mathbf{r}) = \hat{x}\frac{IR}{c}&\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\beta}{k^2}J_1[kR]\cos(\alpha x)\sin(\beta y)\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &-\hat{y}\frac{IR}{c}\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\alpha}{k^2}J_1[kR]\sin(\alpha x)\cos(\beta y)\,\mathrm{e}^{-k|z-b|}\,\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned} \tag{A.4}$$

Appendix B. Definite integrals

The following integrals can be calculated from integral 3.937 of Gradshteyn and Ryzhik (1980):

$$\int_0^{2\pi}\cos[\delta\cos(\theta)]\cos[\gamma\sin(\theta)]\,\mathrm{d}\theta = 2\pi J_0\left[\sqrt{\delta^2+\gamma^2}\right] \tag{B.1}$$

$$\int_0^{2\pi} \sin(\theta)\cos[\delta\cos(\theta)]\sin[\gamma\sin(\theta)]\,\mathrm{d}\theta = \frac{2\pi\gamma}{\sqrt{\delta^2+\gamma^2}} J_1\left[\sqrt{\delta^2+\gamma^2}\right] \tag{B.2}$$

$$\int_0^{2\pi} \cos^2(\theta)\cos[\delta\cos(\theta)]\cos[\gamma\sin(\theta)]\,\mathrm{d}\theta$$
$$= \pi J_0\left[\sqrt{\delta^2+\gamma^2}\right] + \frac{\pi(\gamma^2-\delta^2)}{\delta^2+\gamma^2} J_2\left[\sqrt{\delta^2+\gamma^2}\right] \tag{B.3}$$

$$\int_0^{2\pi} \cos(\theta)\sin(\theta)\sin[\delta\cos(\theta)]\sin[\gamma\sin(\theta)]\,\mathrm{d}\theta = \frac{2\pi\delta\gamma}{\delta^2+\gamma^2} J_2\left[\sqrt{\delta^2+\gamma^2}\right]. \tag{B.4}$$

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