Some Recent Developments in the Analytic Hierarchy Process

by

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Focus of Presentation

Celebrating nearly 30 years of AHP-based decision making

AHP overview

Linear programming models for AHP

Computational experiments

Conclusions
Number of AHP Papers in EJOR (last 20 years)
AHP Articles in Press at EJOR

- Solving multiattribute design problems with the analytic hierarchy process and conjoint analysis: An empirical comparison

- Understanding local ignorance and non-specificity within the DS/AHP method of multi-criteria decision making

- Phased multicriteria preference finding

- Interval priorities in AHP by interval regression analysis

- A fuzzy approach to deriving priorities from interval pairwise comparison judgments

- Representing the strengths and directions of pairwise comparisons
A Recent Special Issue on AHP

- Guest Editors: B. Golden and E. Wasil
- Articles
  - Celebrating 25 years of AHP-based decision making
  - Decision counseling for men considering prostate cancer screening
  - Visualizing group decisions in the analytic hierarchy process
  - Using the analytic hierarchy process as a clinical engineering tool to facilitate an iterative, multidisciplinary, microeconomic health technology assessment
  - An approach for analyzing foreign direct investment projects with application to China’s Tumen River Area development
  - On teaching the analytic hierarchy process
A Recent Book on AHP


Authors: N. Bhushan and K. Rai

Contents

Part I. Strategic Decision-Making and the AHP
   1. Strategic Decision Making
   2. The Analytic Hierarchy Process

Part II. Strategic Decision-Making in Business
   3. Aligning Strategic Initiatives with Enterprise Vision
   4. Evaluating Technology Proliferation at Global Level
   5. Evaluating Enterprise-wide Wireless Adoption Strategies
   6. Software Vendor Evaluation and Package Selection
   7. Estimating the Software Application Development Effort at the Proposal Stage
Part III. Strategic Decision-Making in Defense and Governance

8. Prioritizing National Security Requirements
9. Managing Crisis and Disorder
10. Weapon Systems Acquisition for Defense Forces
11. Evaluating the Revolution in Military Affairs (RMA) Index of Armed Forces
12. Transition to Nuclear War
AHP and Related Software

- **Expert Choice (Forman)**
  EC Resource Aligner combines optimization with AHP to select the optimal combination of alternatives or projects subject to a budgetary constraint.

- **Criterium DecisionPlus (Hearne Scientific Software)**

- **HIPRE 3+ (Systems Analysis Laboratory, Helsinki)**

- **Web-HIPRE**
  The first web-based multiattribute decision analysis tool.

- **Super Decisions (Saaty)**
  This software implements the analytic network process (decision making with dependence and feedback).
AHP Overview

- Analysis tool that provides insight into complex problems by incorporating qualitative and quantitative decision criteria.
- Hundreds of published applications in numerous different areas.
- Combined with traditional OR techniques to form powerful “hybrid” decision support tools.
- Four step process.
Step 1. Decompose the problem into a hierarchy of interrelated decision criteria and alternatives

Hierarchy with P Levels
The Analytic Hierarchy Process

Illustrative example

Level 1: Focus

Level 2: Criteria
- Scientific
- Economic
- Political

Level 3: Subcriteria
- Statewide
- Local

Level 4: Alternatives
- Close
- Restricted Access
- Open Access

Partial Hierarchy: Management of a Fishery
The Analytic Hierarchy Process

Step 2. Use collected data to generate pairwise comparisons at each level of the hierarchy

Illustrative Example

<table>
<thead>
<tr>
<th></th>
<th>Scientific</th>
<th>Economic</th>
<th>Political</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scientific</td>
<td>1</td>
<td>$a_{SE}$</td>
<td>$a_{SP}$</td>
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<tr>
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<td>1</td>
<td>$a_{EP}$</td>
</tr>
<tr>
<td>Political</td>
<td>$1/a_{SP}$</td>
<td>$1/a_{EP}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Pairwise Comparison Matrix: Second Level
The Analytic Hierarchy Process

- Compare elements two at a time

- Generate the $a_{SE}$ entry
  - With respect to the overall goal, which is more important – the scientific or economic factor – and how much more important is it?
  - Number from 1/9 to 9
  - Positive reciprocal matrix
The Analytic Hierarchy Process

Illustrative Example

<table>
<thead>
<tr>
<th></th>
<th>Scientific</th>
<th>Economic</th>
<th>Political</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scientific</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Economic</td>
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<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Political</td>
<td>1/5</td>
<td>1/2</td>
<td>1</td>
</tr>
</tbody>
</table>

AHP provides a way of measuring the consistency of decision makers in making comparisons

Decision makers are not required or expected to be perfectly consistent
The Analytic Hierarchy Process

Step 3. Apply the eigenvalue method (EM) to estimate the weights of the elements at each level of the hierarchy

The weights for each matrix are estimated by solving

\[ A \cdot \hat{w} = \lambda_{\text{MAX}} \cdot \hat{w} \]

where

- A is the pairwise comparison matrix
- \( \lambda_{\text{MAX}} \) is the largest eigenvalue of A
- \( \hat{w} \) is its right eigenvector
The Analytic Hierarchy Process

Illustrative Example

<table>
<thead>
<tr>
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<th>Economic</th>
<th>Political</th>
<th>Weights</th>
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<tr>
<td>Economic</td>
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<td>2</td>
<td>.276</td>
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<tr>
<td>Political</td>
<td>1/5</td>
<td>1/2</td>
<td>1</td>
<td>.128</td>
</tr>
</tbody>
</table>

Pairwise comparison matrix: Second level
The Analytic Hierarchy Process

Step 4. Aggregate the relative weights over all levels to arrive at overall weights for the alternatives.
Estimating Weights in the AHP

Traditional method: Solve for \( \hat{w} \) in \( A\hat{w} = \lambda_{\text{MAX}} \hat{w} \)

Alternative approach (Logarithmic Least Squares or LLS): Take the geometric mean of each row and then normalize

Linear Programming approach (Chandran, Golden, Wasil, Alford)

- Let \( \frac{w_i}{w_j} = a_{ij} \varepsilon_{ij} \) \((i, j = 1, 2, \ldots, n)\) define an error \( \varepsilon_{ij} \) in the estimate \( a_{ij} \)
- If the decision maker is perfectly consistent, then \( \varepsilon_{ij} = 1 \) and \( \ln \varepsilon_{ij} = 0 \)
- We develop a two-stage LP approach
Linear Programming Setup

Given: \( A = [a_{ij}] \) is \( n \times n \)

Decision variables
- \( w_i = \) weight of element \( i \)
- \( \varepsilon_{ij} = \) error factor in estimating \( a_{ij} \)

Transformed decision variables
- \( x_i = \ln(\ w_i) \)
- \( y_{ij} = \ln(\ \varepsilon_{ij}) \)
- \( z_{ij} = |y_{ij}| \)
Some Observations

Take the natural log of $w_i / w_j = a_{ij} \varepsilon_{ij}$ to obtain

$$x_i - x_j - y_{ij} = \ln a_{ij}$$

If $a_{ij}$ is overestimated, then $a_{ji}$ is underestimated

- $\varepsilon_{ij} = 1 / \varepsilon_{ji}$
- $y_{ij} = -y_{ji}$

$z_{ij} \geq y_{ij}$ and $z_{ij} \geq y_{ji}$ identifies the element that is overestimated and the magnitude of overestimation

We can arbitrarily set $w_1 = 1$ or $x_1 = \ln (w_1) = 0$ and normalize the weights later
First Stage Linear Program

Minimize \( \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij} \)

subject to

\( x_i - x_j - y_{ij} = \ln a_{ij}, \quad i, j = 1, 2, \ldots, n; \ i \neq j, \)
\( z_{ij} \geq y_{ij}, \quad i, j = 1, 2, \ldots, n; \ i < j, \)
\( z_{ij} \geq y_{ji}, \quad i, j = 1, 2, \ldots, n; \ i < j, \)
\( x_1 = 0, \)
\( x_i - x_j \geq 0, \quad i, j = 1, 2, \ldots, n; \ a_{ij} > 1, \)
\( x_i - x_j \geq 0, \quad i, j = 1, 2, \ldots, n; \ a_{ik} \geq a_{jk} \) for all \( k; \)
\( a_{i q} > a_{j q} \) for some \( q, \)
\( z_{ij} \geq 0, \quad i, j = 1, 2, \ldots, n, \)
\( x_i, y_{ij} \) unrestricted \( i, j = 1, 2, \ldots, n \)

minimize inconsistency

error term def.

degree of overestimation

set one \( w_i \)

element dominance

row dominance
Element and Row Dominance Constraints

ED is preserved if $a_{ij} > 1$ implies $w_i \geq w_j$

EM and LLS do not preserve ED

RD is preserved if $a_{ik} \geq a_{jk}$ for all $k$ and $a_{ik} > a_{jk}$ for some $k$ implies $w_i \geq w_j$

Both EM and LLS guarantee RD

We capture these constraints explicitly in the first stage LP
The Objective Function (OF)

- The OF minimizes the sum of logarithms of positive errors in natural log space.

- In the nontransformed space, the OF minimizes the product of the overestimated errors \((\varepsilon_{ij} \geq 1)\).

- Therefore, the OF minimizes the geometric mean of all errors \(\geq 1\).

- In a perfectly consistent comparison matrix, \(z^* = 0\) (since \(\varepsilon_{ij} = 1\) and \(y_{ij} = 0\) for all \(i\) and \(j\)).
The Consistency Index

- The OF is a measure of the inconsistency in the pairwise comparison matrix.

- The OF minimizes the sum of $n(n-1)/2$ decision variables ($z_{ij}$ for $i < j$).

- The OF provides a convenient consistency index:
  \[
  \text{CI (LP)} = 2 \frac{z^*}{n(n-1)}
  \]

- CI (LP) is the average value of $z_{ij}$ for elements above the diagonal in the comparison matrix.
Multiple Optimal Solutions

The first stage LP minimizes the product of errors $\varepsilon_{ij}$

But, multiple optimal solutions may exist

In the second stage LP, we select from this set of alternative optima, the solution that minimizes the maximum of errors $\varepsilon_{ij}$

The second stage LP is presented next
Second Stage Linear Program

Minimize \( z_{\text{max}} \)

subject to

\[
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij} = z^*,
\]

\( z_{\text{max}} \geq z_{ij}, \quad i, j = 1, 2, \ldots, n; \quad i < j, \)

and all first stage LP constraints

\( z^* \) is the optimal first stage solution value

\( z_{\text{max}} \) is the maximum value of the errors \( z_{ij} \)
Illustrating Some Constraints

Fig. 1. 3 x 3 pairwise comparison matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1/3</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Error term def. constraint \( (a_{12}) \)

\[
x_1 - x_2 - y_{12} = \ln a_{12} = 0.693
\]

Element dominance constraints \((a_{12} \text{ and } a_{13})\)

\[
x_1 - x_2 \geq 0 \text{ and } x_1 - x_3 \geq 0
\]

Row dominance constraints

\[
x_1 - x_2 \geq 0, \quad x_1 - x_3 \geq 0, \text{ and } x_2 - x_3 \geq 0
\]
Advantages of LP Approach

Simplicity
- Easy to understand
- Computationally fast
- Readily available software
- Easy to measure inconsistency

Sensitivity Analysis
- Which $a_{ij}$ entry should be changed to reduce inconsistency?
- How much should the entry be changed?
More Advantages of the LP Approach

Ensures element dominance and row dominance

- Limited protection against rank reversal

Generality

- Interval judgments
- Mixed pairwise comparison matrices
- Group decisions
- Soft interval judgments
Modeling Interval Judgments

In traditional AHP, $a_{ij}$ is a single number that estimates $w_i / w_j$.

Alternatively, suppose an interval $[l_{ij}, u_{ij}]$ is specified.

Let us treat the interval bounds as hard constraints.

Two techniques to handle interval judgments have been presented by Arbel and Vargas:

- Preference simulation
- Preference programming
Preference Simulation

- Sample from each interval to obtain a single $a_{ij}$ value for each matrix entry

- Repeat this $t$ times to obtain $t$ pairwise comparison matrices

- Apply the EM approach to each matrix to produce $t$ priority vectors

- The average of the feasible priority vectors gives the final set of weights
Preference Simulation Drawbacks

This approach can be extremely inefficient when most of the priority vectors are infeasible.

This can happen as a consequence of several tight interval judgments.

How large should $t$ be?

Next, we discuss preference programming.
Preference Programming

It begins with the linear inequalities and equations below:

\[
\begin{align*}
    l_{ij} &\leq w_i/ w_j \leq u_{ij}, & i, j = 1, 2, \ldots, n; & i < j, \\
    \sum_{i=1}^{n} w_i &= 1, \\
    w_i &\geq 0, & i = 1, 2, \ldots, n
\end{align*}
\]

LP is used to identify the vertices of the feasible region.

The arithmetic mean of these vertices becomes the final priority vector.

No attempt is made to find the best vector in the feasible region.
More on the Interval AHP Problem

Fig. 2. 3 x 3 pairwise comparison matrix with lower and upper bounds \([ l_{ij}, u_{ij} ]\) for each entry

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>[5,7]</th>
<th>[2,4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1/7,1/5]</td>
<td>1</td>
<td>[1/3,1/2]</td>
<td></td>
</tr>
<tr>
<td>[1/4,1/2]</td>
<td>[2,3]</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

- Entry \(a_{12}\) is a number between 5 and 7
- The matrix is reciprocal
- Entry \(a_{21}\) is a number between 1/7 and 1/5
- The first stage LP can be revised to handle the interval AHP problem
A New LP Approach for Interval Judgments

Set $a_{ij}$ to the geometric mean of the interval bounds

$$a_{ij} = (l_{ij} \times u_{ij})^{\frac{1}{2}}$$

This preserves the reciprocal property of the matrix.

If we take natural logs of $l_{ij} \leq w_i / w_j \leq u_{ij}$, we obtain

$$x_i - x_j \geq \ln l_{ij}, \quad i, j = 1, 2, \ldots, n; \quad i < j,$$

$$x_i - x_j \leq \ln u_{ij}, \quad i, j = 1, 2, \ldots, n; \quad i < j.$$
Further Notes

When $l_{ij} > 1$, \[ x_i - x_j \geq \ln l_{ij} \Rightarrow x_i - x_j \geq 0 \Rightarrow w_i \geq w_j \]
and behaves like an element dominance constraint

When $u_{ij} < 1$, \[ x_i - x_j \leq \ln u_{ij} \Rightarrow x_i - x_j \leq 0 \Rightarrow w_j \geq w_i \]
and behaves like an element dominance constraint

Next, we formulate the first stage model for handling interval judgments
First Stage Linear Program for Interval AHP

Minimize \[
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij}
\]
subject to
\[
\begin{align*}
x_i - x_j - y_{ij} &= \ln a_{ij}, & i, j = 1, 2, \ldots, n; i \neq j, \\
z_{ij} &\geq y_{ij}, & i, j = 1, 2, \ldots, n; i < j, \\
z_{ij} &\geq y_{ji}, & i, j = 1, 2, \ldots, n; i < j, \\
x_1 &= 0, \\
x_i - x_j &\geq \ln l_{ij}, & i, j = 1, 2, \ldots, n; i < j, \\
x_i - x_j &\leq \ln u_{ij}, & i, j = 1, 2, \ldots, n; i < j, \\
z_{ij} &\geq 0, & i, j = 1, 2, \ldots, n, \\
x_i, y_{ij} &\text{ unrestricted} & i, j = 1, 2, \ldots, n
\end{align*}
\]

minimize inconsistency
	error term def. (GM)

degree of overestimation

set one \( w_i \)

lower bound constraint

upper bound constraint

Note: The second stage LP is as before
Mixed Pairwise Comparison Matrices

Suppose, as above, some entries are single numbers $a_{ij}$ and some entries are intervals $[l_{ij}, u_{ij}]$.

Our LP approach can easily handle this mixed matrix problem.

The first stage LP is nearly the same as for the interval AHP.

We add element dominance constraints, as needed.

$$x_1 - x_3 \geq 0$$
Modeling Group Decisions

Suppose there are $n$ decision makers

Most common approach

- Have each decision maker $k$ fill in a comparison matrix independently to obtain $[a_{k,ij}]$
- Combine the individual judgments using the geometric mean to produce entries $A = [a_{ij}]$ where

$$a_{ij} = [a_{1,ij}^1 	imes a_{2,ij}^2 	imes \ldots 	imes a_{n,ij}^n]^{1/n}$$

- EM is applied to $A$ to obtain the priority vector
Modeling Group Decisions using LP

An alternative direction is to apply the LP approach to mixed pairwise comparison matrices.

We compute interval bounds as below (for $i < j$)

\[
\begin{align*}
    l_{ij} &= \min \{ a^1_{ij}, a^2_{ij}, \ldots, a^n_{ij} \} \\
    u_{ij} &= \max \{ a^1_{ij}, a^2_{ij}, \ldots, a^n_{ij} \}
\end{align*}
\]

If $l_{ij} = u_{ij}$, we use a single number, rather than an interval.

If $n$ is large, we can eliminate the high and low values and compute interval bounds or a single number from the remaining $n - 2$ values.
Suppose we have interval constraints, but they are too tight to admit a feasible solution.

We may be interested in finding the “closest-to-feasible” solution that minimizes the first stage and second stage LP objective functions.

Imagine that we multiply each upper bound by a stretch factor $\lambda_{ij} \geq 1$ and that we multiply each lower bound by the inverse $1/\lambda_{ij}$.

The geometric mean given by $a_{ij} = (l_{ij} u_{ij})^{\frac{1}{2}} = (l_{ij}/\lambda_{ij} \times u_{ij} \lambda_{ij})^{\frac{1}{2}}$ remains the same as before.
Setup for the Phase 0 LP

Let $g_{ij} = \ln (\lambda_{ij})$, which is nonnegative since $\lambda_{ij} \geq 1$.

We can now solve a Phase 0 LP, followed by the first stage and second stage LPs.

The Phase 0 objective is to minimize the product of stretch factors or the sum of the natural logs of the stretch factors.

If the sum is zero, the original problem was feasible.

If not, the first and second stage LPs each include a constraint that minimally stretches the intervals in order to ensure feasibility.
Stretched Upper Bound Constraints

- Start with \( w_i / w_j \leq u_{ij} \lambda_{ij} \)

- Take natural logs to obtain

\[
x_i - x_j \leq \ln(u_{ij}) + \ln(\lambda_{ij})
\]

\[
x_i - x_j \leq \ln(u_{ij}) + g_{ij}
\]

\[
x_i - x_j - g_{ij} \leq \ln(u_{ij})
\]

- Stretched lower bound constraints are generated in the same way
The Phase 0 LP

Minimize \[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} g_{ij} \]

\[ x_i – x_j + g_{ij} \geq \ln (l_{ij}), \quad i, j = 1, 2, \ldots, n; \quad i < j, \]

\[ x_i – x_j – g_{ij} \leq \ln (u_{ij}), \quad i, j = 1, 2, \ldots, n; \quad i < j, \]

error term def. (GM),

degree of overestimation,

set one \( w_i \),

\[ z_{ij}, g_{ij} \geq 0 \quad i, j = 1, 2, \ldots, n, \]

\( x_i, y_{ij} \) unrestricted \( i, j = 1, 2, \ldots, n \)
Two Key Points

We have shown that our LP approach can handle a wide variety of AHP problems

- Traditional AHP
- Interval judgments
- Mixed pairwise comparison matrices
- Group decisions
- Soft interval judgments

As far as we know, no other single approach can handle all of the above variants
Computational Experiment: Inconsistency

We see that element 4 is less important than element 6

We expect to see $w_4 \leq w_6$

Upon closer examination, we see $a_{46} = a_{67} = a_{74} = \frac{1}{2}$

We expect to see $w_4 = w_6 = w_7$

---

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>1</th>
<th>4</th>
<th>2</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
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<td>1</td>
<td>1/3</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>1</td>
<td>1/5</td>
<td>1</td>
<td>1/2</td>
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<td>1/2</td>
<td></td>
</tr>
<tr>
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<td>1/2</td>
<td>1/3</td>
<td>1/2</td>
<td>1/2</td>
<td>2</td>
<td>1</td>
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</tbody>
</table>

Fig. 4. Matrix 1
The Impact of Element Dominance

Table 1
Priority vectors for Matrix 1

<table>
<thead>
<tr>
<th>Weight</th>
<th>EM</th>
<th>LLS</th>
<th>Second-stage LP model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RD</td>
<td>RD</td>
<td>ED and RD</td>
</tr>
<tr>
<td>$w_1$</td>
<td>0.291</td>
<td>0.312</td>
<td>0.303</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.078</td>
<td>0.073</td>
<td>0.061</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.300</td>
<td>0.293</td>
<td>0.303</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.064</td>
<td>0.064</td>
<td>0.061</td>
</tr>
<tr>
<td>$w_5$</td>
<td>0.159</td>
<td>0.157</td>
<td>0.152</td>
</tr>
<tr>
<td>$w_6$</td>
<td>0.051</td>
<td>0.044</td>
<td>0.061</td>
</tr>
<tr>
<td>$w_7$</td>
<td>0.058</td>
<td>0.057</td>
<td>0.061</td>
</tr>
</tbody>
</table>

ED: Element Dominance, RD: Row Dominance
Another Example of Element Dominance

Fig. 5. Matrix 2

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>8</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1</td>
<td>1/1.5</td>
<td>7</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>1/2.5</td>
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<td>1</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
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<td>1/7</td>
<td>1/5</td>
<td>1</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>1/5</td>
<td>1/5</td>
<td>1/3</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The decision maker has specified that $w_2 \leq w_3$

EM and LLS violate this ED constraint

As with Matrix 1, the weights from EM, LLS, and LP are very similar
## Computational Results for Matrix 2

### Table 2

Priority vectors for Matrix 2

<table>
<thead>
<tr>
<th>Weight</th>
<th>EM</th>
<th>LLS</th>
<th>Second-stage LP model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RD</td>
<td>RD</td>
<td>ED and RD</td>
</tr>
<tr>
<td>$w_1$</td>
<td>0.419</td>
<td>0.422</td>
<td>0.441</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.242</td>
<td>0.239</td>
<td>0.221</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.229</td>
<td>0.227</td>
<td>0.221</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.041</td>
<td>0.041</td>
<td>0.044</td>
</tr>
<tr>
<td>$w_5$</td>
<td>0.070</td>
<td>0.071</td>
<td>0.074</td>
</tr>
</tbody>
</table>

ED: Element Dominance, RD: Row Dominance
Computational Experiment: Interval AHP

Fig. 6. Matrix 3

|     | [2,5] | [2,4] | [1,3] | [1/5,1/2] | 1 | [1,3] | [1,2] | [1/4,1/2] | [1/3,1] | 1 | [1/2,1] | [1/3,1] | [1/2,1] | [1,2] | 1 |
|-----|-------|-------|-------|-----------|----|-------|-------|-----------|-------|----|-------|-------|-------|-------|----|---|
| 1   |       |       |       |           |    |       |       |           |       |    |       |       |       |       |    |   |

Table 3
Priority vectors for Matrix 3

<table>
<thead>
<tr>
<th>Weight</th>
<th>Preference simulationa</th>
<th>Preference programminga</th>
<th>Second-stage LP model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Minimum</td>
<td>Average</td>
<td>Maximum</td>
</tr>
<tr>
<td>$w_1$</td>
<td>0.369</td>
<td>0.470</td>
<td>0.552</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.150</td>
<td>0.214</td>
<td>0.290</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.093</td>
<td>0.132</td>
<td>0.189</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.133</td>
<td>0.184</td>
<td>0.260</td>
</tr>
</tbody>
</table>

a Results from Arbel and Vargas
Computational Experiment with a Mixed Pairwise Comparison Matrix

**Fig. 7. Matrix 4**

<table>
<thead>
<tr>
<th></th>
<th>[2,4]</th>
<th>4</th>
<th>[4.5,7.5]</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>[1/5,1/3]</td>
</tr>
<tr>
<td>[1/4,1/2]</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td></td>
<td>1</td>
<td>[1,2]</td>
<td>1/2</td>
</tr>
<tr>
<td>[1/7,5,1/4.5]</td>
<td>1/2</td>
<td>[1/2,1]</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>1</td>
<td>[3,5]</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

- We converted every interval entry into a single $a_{ij}$ entry by taking the geometric mean of the lower bound and upper bound.

- We applied EM to the resulting comparison matrix.

- We compared the EM and LP results.
Computational Results for Matrix 4

Table 4
Priority vectors for Matrix 4

<table>
<thead>
<tr>
<th>Weight</th>
<th>EM</th>
<th>Second-stage LP model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0.377</td>
<td>0.413</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.117</td>
<td>0.103</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.116</td>
<td>0.103</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.076</td>
<td>0.071</td>
</tr>
<tr>
<td>$w_5$</td>
<td>0.314</td>
<td>0.310</td>
</tr>
</tbody>
</table>

We point out that the weights generated by EM violate one of the four interval constraints.

The interval [1/5, 1/3] is violated.
Group AHP Experiment

Four graduate students were given five geometric figures (from Gass)

They were asked to compare (by visual inspection) the area of figure $i$ to the area of figure $j$ ($i < j$)

Lower and upper bounds were determined, as well as geometric means

Since $l_{34} = u_{34} = 4.00$, we use a single number for $a_{34}$

Otherwise, we have interval constraints
# Geometry Experiment Results

**Table 5**

Priority vectors for geometry experiment

<table>
<thead>
<tr>
<th>Weight</th>
<th>EM</th>
<th>LLS</th>
<th>Second-stage LP model</th>
<th>Actual geometric areas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0.272</td>
<td>0.272</td>
<td>0.277</td>
<td>0.273</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.096</td>
<td>0.096</td>
<td>0.095</td>
<td>0.091</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.178</td>
<td>0.178</td>
<td>0.172</td>
<td>0.182</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.042</td>
<td>0.042</td>
<td>0.041</td>
<td>0.045</td>
</tr>
<tr>
<td>$w_5$</td>
<td>0.412</td>
<td>0.412</td>
<td>0.414</td>
<td>0.409</td>
</tr>
</tbody>
</table>

The three priority vectors and the actual geometric areas (normalized to sum to one) are presented above.

They are remarkably similar.
Computational Experiment with Soft Intervals

We observe that several intervals are quite narrow

We apply Phase 0 and the two-stage LP approach
Soft Interval (Matrix 5) Results

The optimal stretch factors are

\[ \lambda_{12} = 1.2248, \quad \lambda_{23} = 1.0206, \]

\[ \lambda_{13} = \lambda_{14} = \lambda_{24} = \lambda_{34} = 1 \]

The \( a_{12} \) and \( a_{23} \) intervals stretch from

- \([2,5]\) to \([1.6329, 6.124]\)
- \([2.5,3]\) to \([2.4495, 3.0618]\)

The optimal weights are

\[ w_1 = 0.4233, \quad w_2 = 0.2592, \quad w_3 = 0.1058, \quad w_4 = 0.2116 \]
Conclusions

- We have presented a compact LP approach for estimating priority vectors in the AHP.

- In general, the weights generated by EM, LLS, and our LP approach are similar.

- The LP approach has several advantages over EM and LLS:
  - LPs are easy to understand
  - Sensitivity analysis
  - Our measure of inconsistency is intuitively appealing
  - Ensures ED and RD conditions
  - Our approach is more general
The End (Really)

The LP approach can handle a wide variety of AHP problems

- Traditional AHP
- Interval entries
- Mixed entries
- Soft intervals
- Group AHP

We hope to explore extensions and new applications of this approach in future research

Thank you for your patience