Turbulent Vorticity Transport in Three Dimensions\footnote{This study was initiated while the author was Professor Associé a l'Université Claude Bernard, Lyon, France. Additional support was provided by DOE Project No. DE-FG05-85ER13313.A000, the Ford Motor Co., and the Naval Research Laboratory through an ASEE/Navy fellowship.}

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Abstract. The nature of vorticity transport in three-dimensional turbulent flow is investigated using data from a direct numerical simulation of channel flow. An extension of Taylor's (1932) vorticity transport theory is derived by a formal Lagrangian analysis in which care is taken to include nongradient transport effects associated with vortex stretching and shearing. A comparison is made between the predicted and exact forms of the nine components of the vorticity transport tensor as calculated from the simulation data. It is found that the qualitative agreement between the theory and the computed fluxes is excellent. The nongradient terms make a significant contribution near the boundary as an apparent consequence of the dynamical importance of coherent vortical structures. A model for the Reynolds shear stress consisting of a gradient term together with a nonlocal integral expression representing the effect of the pressure field on transport is extracted from the vorticity transport law. Computations reveal that a gradient vorticity transport model is acceptable only at a distance from the wall, while the opposite is true in the case of momentum transport. These results agree with much earlier studies of G.I. Taylor.

1. Introduction

The tendency of turbulent flows to transport vorticity through random motions is one of their most fundamental processes. Thus, an understanding of the physics underlying the vorticity flux tensor has great intrinsic importance as a means of conceptualizing the dynamics of turbulent flow. It is also of significant practical worth since one of the principal avenues by which the turbulence closure problem can be formulated is through the vorticity transport form of the averaged momentum equation. This point was first made by Taylor (1915) in a study of the two-dimensional atmospheric boundary layer in which he modeled vorticity transport in the course of closing the momentum equation. Taylor assumed that the chief physical mechanism responsible for vorticity transport lay in gradient diffusion brought on by the turbulent fluctuations. The rationale for this viewpoint came from his observation that in two-dimensional inviscid flow fluid particles preserve vorticity along their paths, thus satisfying one of the major requirements of a gradient model. In three-dimensional flows, however, where vortex stretching and shearing are present to alter the vorticity of fluid particles, it is unlikely that gradient diffusion can be an entirely satisfactory characterization of the physics of vorticity transport. Indeed,
in several calculations Taylor (1935, 1937) saw that a gradient transport model was deficient close to solid boundaries where three-dimensional effects can be assumed to be largest. From this it may be inferred that nongradient diffusion has a significant role to play in the dynamics of vorticity transport.

Beyond the inadequacies of the gradient vorticity transport model described by Taylor, several more recent studies have been more precise in explaining why such a model is unlikely to account for all aspects of turbulent diffusion. In particular, Corrsin (1974) showed that the length scale of turbulent transport is often greater than that of linear variation of the mean field, thus violating a condition necessary for the applicability of a gradient model. Tennekes and Lumley (1972) have pointed out that gradient transport models are not easily reconciled with the major influence that coherent structures can be expected to have on turbulent diffusion near solid walls. A specific criticism of gradient vorticity transport was given by Stewart and Thomson (1977) who argued that it had to be incorrect on general principles. In particular, they believed that it was incompatible with the conservation of momentum in certain simple flow fields. Marshall (1981), however, later showed that this conclusion was unwarranted. Gradient vorticity transport models continue to be used today, though principally in the context of closure schemes for the potential vorticity, a quantity which appears in many treatments of meteorological flows (Ivchenko and Klepikov 1985).

In his three-dimensional vorticity-transport theory, Taylor (1932) attempted to elucidate the nongradient flow processes which contribute to the turbulent flux of vorticity. He developed a formal Lagrangian procedure for analyzing the vorticity transport correlation which yielded concrete mathematical forms for the nongradient diffusion effects. Unfortunately, these expressions were in a form which made their physical interpretation quite difficult. Moreover, they were not useful in a computational sense as well so that Taylor was forced to drop them from consideration when developing a practical closure scheme based on vorticity-transport theory.

Since the time of Taylor's investigations into the nature of vorticity transport only a few studies have attempted to explore this process further. Among these Chorin (1974, 1975) derived a gradient vorticity-transport model for two-dimensional turbulence as one aspect of a novel closure scheme. In this, the mixing length was tied to the product of a time scale and the fluctuating velocity field. The result was a computable representation of the eddy viscosity. The present author (Bernard, 1980), in a refinement of Chorin's approach, developed a formalized Lagrangian analysis of vorticity transport in two dimensions which allowed for the determination of higher-order effects. It also revealed that the time scale appearing in the eddy viscosity must be a Lagrangian integral scale. Rhines and Holland (1979) have also used a Lagrangian integration technique to develop a gradient transport law for potential vorticity. More recently, Bernard and Berger (1982) introduced a Lagrangian approach for the analysis of three-dimensional vorticity transport which yielded specific expressions for nongradient phenomena arising from the effects of vortex stretching and shearing. However, the physics of these new terms was not studied nor were they incorporated in several applications of the transport law in the context of a closure scheme (Bernard, 1981, 1987; Raul, 1988).

The nature of vorticity transport can, in principle, be studied through experimental techniques capable of measuring simultaneously velocity and vorticity (Balint et al., 1988). In fact, some preliminary calculations of vorticity transport through this source have been made (Balint et al., 1987) in a boundary-layer flow. This data, however, is for the region away from the wall where it is expected that gradient diffusion is the dominant characteristic of vorticity transport. A potentially more fruitful source of information about the behavior of vorticity transport, particularly in the region adjacent to solid boundaries, is through direct numerical simulation studies. In these, the correlations between velocity and vorticities are readily obtainable as is evident from a recent study of the fluctuating helicity field (Rogers and Moin, 1987).

The intent of this study is to examine the physics of turbulent vorticity transport as revealed by the use of direct numerical simulation data of channel flow in conjunction with a Lagrangian analysis of the transport correlation. To accomplish this a turbulent vorticity transport law is first derived following the analytical approach given previously (Bernard and Berger, 1982) but with particular care to include all important nongradient transport effects. The validity of the derived transport law is verified by comparison of its predictions of the nine components of the vorticity-transport tensor with the data from the direct numerical simulation of channel flow. These tests strongly indicate that the transport law captures the major portion of the physics of vorticity transport. In this, the nongradient
terms are seen to make an essential contribution, primarily near the boundary. To examine the physics of the vorticity flux in the channel, the computed forms taken by the transport correlations are related to the physical processes associated with the analytical terms. By this step it is revealed that the trends in the vorticity fluxes in channel flow are consistent with current models of the coherent vortical structures of the wall region.

The Lagrangian transport analysis described below leads to tractable expressions for the non-gradient transport effects, in contrast to the approach pursued by Taylor. However, it will also be shown that the present results are obtainable by a relatively minor alteration to Taylor's original vorticity-transport theory. Thus, the developments of the current study are in a real sense an outgrowth of Taylor's pioneering work. In the same way that Taylor used a vorticity-transport model to close the momentum equation, it is natural to explore the implications of the present result in this regard as well. Consequently, numerical tests of the closed form of the momentum equation implied by the derived transport law have been undertaken. These show it to yield good predictions of the mean velocity field. The Reynolds stress closure which is implied by these results is also deduced. This contains an explicit nonlocal, nongradient term representing the effect of pressure forces on momentum transport. Calculation of the gradient contribution to the momentum flux shows it to be a good representation of transport only near the boundary. Examination of the gradient term in the derived vorticity-transport law indicates that it is an acceptable approximation only at a distance from the wall. Both of these conclusions agree with the much earlier studies of Taylor (1935, 1937).

The next section presents a formal derivation of the vorticity-transport law in which its connection to Taylor's vorticity-transport theory is indicated. Following this, Section 3 describes the implications of the theory insofar as channel flow is concerned and the predicted fluxes are verified using simulation data. A discussion as to the nature of vorticity transport in channel flow is presented in Section 4 in which attention is given to the relationship of the transport law to the vortex structure of the wall region of turbulent flow. Further elaboration of the transport law for the most general three-dimensional mean flow field is provided in Section 5 and finally, in the last section, conclusions are given.

2. Analysis of Vorticity Transport

It is desired to account for the sources of correlation in the vorticity-transport tensor $u'\omega'$ representing the turbulent flux of $\omega$ in the direction $x_1$ where $u_i$ and $\omega_j$ are the velocity and vorticity fluctuation vectors, respectively. The superscript "a" in this and subsequent expressions is meant to denote quantities evaluated at a given position $a$ in the flow at a given time $t_0$. The methods previously developed by Taylor (1932) and Bernard and Berger (1982) in modeling $u'\omega'$ involve substituting for $\omega_j$ its representation in terms of one or the other of two alternative Lagrangian identities. The correlation that $u'_i$ may have with $\omega'_j$ is reflected in its averaged product with the terms appearing in the Lagrangian expansions of $\omega'_j$. In the following, Taylor's approach is first briefly described and then followed by a discussion of the technique developed by the present author.

Taylor started his vorticity-transport analysis from the general three-dimensional inviscid relation

$$\Omega^j = \Omega^{ia} \frac{\partial a_j}{\partial b_k}$$

(1)

given by Lamb (1945) from an original work by Cauchy (1827). Equation (1) connects the values of the vorticity vector, $\Omega$, at the end points $a$ and $b$ of a fluid particle path $x(b, t)$ for which $x(b, t_0 - \tau) = b$ and $x(b, t_0) = a$, where $\tau > 0$ is a small time interval. The superscript "b" refers to quantities evaluated at the point $b$ which varies randomly from realization to realization of the flow field. Equation (1) differs in notation from the formula used by Taylor both in employing index notation as well as in using the symbols $a$ and $b$ to denote what Taylor referred to as $x$ and $a$, respectively. Substituting the Reynolds decompositions $\Omega^j = \Omega^j + \omega^j$ and $\Omega^j = \Omega^j + \omega^j$ into equation (1), where the overbar denotes ensemble averaging, leads to the representation formula

$$\omega^j = \Omega^k \frac{\partial a_j}{\partial b_k} + \omega^k \frac{\partial a_j}{\partial b_k}.$$  

(2)
At this point Taylor summarily dropped the last term in equation (2), replaced $\Omega^2$ by its Taylor series expansion about $a$, and substituted the result in $u_i\omega^f_j$ yielding

$$u_i\omega^f_j = -u_i(a_m - b_m)\frac{\partial a_j}{\partial b_k} + \Omega_k\frac{\partial a_j}{\partial b_k}.$$  (3)

In this expression terms of $O(\tau^2)$ have been dropped and quantities are assumed to be evaluated at point $a$ if not indicated otherwise. The coefficient $u_i(\partial a_j/\partial b_k)$ of the nongradient term in equation (3) Taylor viewed as intractable, so that in later applications of the transport law the nongradient term in equation (3) was omitted.

The alternative representation of $\omega^f_j$ which was used by Bernard and Berger (1982) in modeling vorticity transport is obtained by integrating the three-dimensional vorticity transport equation

$$\frac{\partial \Omega_j}{\partial t} + U_k \frac{\partial \Omega_j}{\partial x_k} = \Omega_k \frac{\partial U_j}{\partial x_k} + \frac{1}{Re} \nabla^2 \Omega_j,$$

along the path $x(b, t)$. Here, $Re$ is an appropriate Reynolds number and $U_i$ is the velocity vector. The integration yields

$$\Omega_j^b - \Omega^0_j = \int_{t_0 - \tau}^{t_0} \Omega_k \frac{\partial U_j}{\partial x_k} ds + \frac{1}{Re} \int_{t_0 - \tau}^{t_0} \nabla^2 \Omega_j ds,$$  (4)

where the variables inside the integrals are understood to be evaluated on the fluid particle path at time $s$. Substituting Reynolds decompositions of the vorticity vector into this equation gives

$$\omega^f_j = \omega^f_j + (\Omega^0_j - \Omega^0_j) + \Omega_k \frac{\partial U_j}{\partial x_k} + \frac{1}{Re} \Omega_k \nabla^2 \Omega_j ds,$$  (5)

which should be contrasted with equation (2). In this Lagrangian decomposition, $\omega^f_j$ is written as the sum of the vorticity fluctuation at $b$, the change in the mean vorticity field between $b$ and $a$, the cumulative vorticity stretching and shearing along the particle path given by the third term on the right-hand side, and finally, in the last term, the cumulative viscous diffusion of vorticity. Before deriving the transport law implied by equation (5) it is of interest to demonstrate that a relation nearly identical to it can be extracted from equation (2) through application of a simple identity. Thus, in effect, both of these relations lead to the same transport law.

In particular, integrating the identity

$$\frac{dx}{dt}(b, t) = U(x(b, t), t),$$

from $t_0 - \tau \rightarrow t_0$, it is found that

$$a_j = b_j + \int_{t_0 - \tau}^{t_0} U_j(x(b, s), s) ds.$$  

Differentiation then gives

$$\frac{\partial a_j}{\partial b_k} = \delta_{jk} + \int_{t_0 - \tau}^{t_0} \frac{\partial U_j}{\partial x_l} \frac{\partial x_l}{\partial b_k} ds,$$

where $\delta_{jk}$ is the Kronecker delta function. Substitution of this into equation (2) gives

$$\omega^f_j = \omega^f_j + (\Omega^0_j - \Omega^0_j) + \Omega_k \frac{\partial U_j}{\partial x_k} + \omega^f_k \frac{\partial U_j}{\partial x_k} \frac{\partial x_l}{\partial b_k} ds + \omega^f_j \int_{t_0 - \tau}^{t_0} \frac{\partial U_j}{\partial x_l} \frac{\partial x_l}{\partial b_k} ds,$$  (6)

in which the last two terms correspond to the second to last term in equation (5). Complete equivalence of equations (5) and (6) is apparently only a matter of appending the viscous term to the latter. It will become clear below that the analysis of $u_i\omega^f_j$ proceeding from either equation (5) or (6) will lead to essentially the same result.

Continuing with the main line of development, it follows from substitution of equation (5) into the vorticity-transport correlation that

$$u_i\omega^2 = u_i\omega^f_j + u_i(\Omega^0_j - \Omega^0_j) + \int_{t_0 - \tau}^{t_0} u_i \Omega_k \frac{\partial U_j}{\partial x_k} ds + \Phi^0_j,$$  (7)
accounts for viscous effects on turbulent vorticity transfer. Equation (7) must hold for all values of \( \tau \).

In particular, when \( \tau \) is sufficiently large, say \( \tau > \tau_1 \), the first term on the right-hand side will be negligible as a consequence of the essential randomizing nature of turbulent flow. This supposition is supported by recent numerical calculations using Lagrangian particle path data (Bernard et al. 1989a, b) which have shown that the related correlation, \( u'_{ij}u'_{ij} \), does indeed approach zero as \( \tau \) increases.

The integrands \( u'_{ij}(\partial U_j/\partial x_k) \) and \( u'_{ij}^2 \Omega_{ij} \) appearing in the last two terms in equation (7) involve the correlation of \( u'_{ij} \) with quantities evaluated at time \( s \), where \( t_0 - \tau \leq s \leq t_0 \). For \( \tau \) greater than, say, \( \tau_2 \), it can be expected that these correlations will approach zero within the interval of definition of \( s \). In this case, the two integral terms in equation (7) will achieve constant values for \( \tau > \tau_2 \). It is thus demonstrated that all of the terms in equation (7), except possibly \( u'_{ij}(\Omega_j^i - \Omega_j^i) \), will be constant once \( \tau > \max(\tau_1, \tau_2) \). However, \( u'_{ij}(\Omega_j^i - \Omega_j^i) \) must also be \( \tau \) independent if this is true of the other terms. Thus it is seen that equation (7) provides a fundamental decomposition of the vorticity transport correlation into component physical processes.

The second term on the right-hand side of equation (7) accounts for transport arising from the random displacement of fluid particles carrying the average vorticity of their initial position to their final destination at time \( t_0 \). It may be expected that the primary contribution to this process is from gradient transport since the sign of \( u'_{ij} \) will determine whether a fluid particle has been traveling from up or down the \( \Omega_j^i \) gradient. The gradient term may be extracted formally by first expanding the difference, \( \Omega_j^i - \Omega_j^i \), in a Taylor series about point \( a \) yielding

\[
\Omega_j^i - \Omega_j^i = -\int_{t_0-\tau}^{t_0} U_k(s) \frac{\partial \Omega_j^i}{\partial x_k} \, ds + \frac{1}{2} \int_{t_0-\tau}^{t_0} U_k(s) \frac{\partial^2 \Omega_j^i}{\partial x_k \partial x_m} \theta, \tag{8}
\]

where \( \theta \) denotes a point between \( b \) and \( a \). Note that terms reflecting the possible time dependence of \( \Omega_j^i \) are not indicated here since they will make no contribution to \( u'_{ij}(\Omega_j^i - \Omega_j^i) \). By employing equation (8) it is found that

\[
u'_{ij}(\Omega_j^i - \Omega_j^i) = -\int_{t_0-\tau}^{t_0} R_{ik}(s) \frac{\partial \Omega_j^i}{\partial x_k} + \Phi_{ij}, \tag{9}
\]

where

\[
R_{ik}(s) \equiv u_{ik}(t_0) u_{ik}(t_0 + s)
\]

is a Lagrangian correlation function, \( u_{ik}(t_0 + s) \) is shorthand for \( u_{ik}(x(b, t_0 + s), t_0 + s) \), and

\[
\Phi_{ij} \equiv \int_{t_0-\tau}^{t_0} ds \int_s^{t_0} dr \, u'_{ik}(r) \frac{\partial U_j^i}{\partial x_n} \frac{\partial \Omega_j^i}{\partial x_k} + \frac{1}{2} \int_{t_0-\tau}^{t_0} ds \int_s^{t_0} dr \, u'_{ik}(s) U_m(r) \frac{\partial^2 \Omega_j^i}{\partial x_k \partial x_m} \theta.
\]

is composed of the higher-order effects which remain after extracting the gradient term. The two terms composing \( \Phi_{ij} \) have their respective origins in the two terms on the right-hand side of equation (8). The symbol \( \theta \) denotes a point on the particle path between time \( s \) and \( t_0 \), which will be different for each realization of the flow field. By the same reasoning as used above it may be concluded that each of the terms into which \( u'_{ij}(\Omega_j^i - \Omega_j^i) \) has been subdivided in equation (9) will be independent of \( \tau \), once \( \tau \) is greater than a critical value.

A decomposition similar to that in equation (9) for the stretching and shearing term in equation (7) may be devised in the form

\[
\int_{t_0-\tau}^{t_0} u'_{ij}(\Omega_j^i \partial U_j^i/\partial x_k) \, ds = -\int_{t_0-\tau}^{t_0} S_{ijk}(s) \, ds \Omega_k + \Phi_{ij}, \tag{10}
\]

where

\[
S_{ijk}(s) \equiv u_{ij}(t_0) \frac{\partial \Omega_j^i}{\partial x_k}(t_0 + s),
\]

and \( \Phi_{ij} \) represents several higher-order terms similar in form to those contained in \( \Phi_{ij} \), as well as a
term containing the uncorrelated factors \( u^e_i \) and \( \omega^e_j \). The formal derivation of equation (10) can be accomplished by either applying integration by parts to the integral on the left-hand side, as was done previously (Bernard and Berger, 1982), or else through a manipulation based on the final two terms in equation (6). Either approach leads to the same end. The first term on the right-hand side of equation (10) is a first-order vorticity stretching and shearing term, which will turn out to be an important source of nongradient vorticity transport. As in the previous cases, the terms in equation (10) will lose their \( \tau \) dependence once the integration interval is large enough.

Assembling the previous results it is found that a decomposition of the vorticity transport correlation has been derived in the form

\[
\overline{u_i \omega_j} = \overline{u_i^e \omega_j^e} - \int_{-\tau}^{0} R_{ijk}(s) \, ds \frac{\partial \overline{\omega_k}}{\partial x_k} + \int_{-\tau}^{0} S_{ijk}(s) \, ds \overline{\omega_k} + \Phi_{ij}^1 + \Phi_{ij}^2 + \Phi_{ij}^3
\]

which has the property that each of its terms remains fixed once \( \tau \) exceeds a critical value. For the purposes of the present study, the viscous term, \( \Phi_{ij}^2 \), which is \( O(\tau/R) \), as well as the two second-order remainder terms \( \Phi_{ij}^2 \) and \( \Phi_{ij}^3 \), are assumed to be of less importance to vorticity transport than the two principal first-order effects indicated in the equation. Consequently, the transport analysis which follows focuses exclusively on the properties of the transport law as given in the form

\[
\overline{u_i \omega_j} = - \int_{-\tau}^{0} R_{ijk}(s) \, ds \frac{\partial \overline{\omega_k}}{\partial x_k} + \int_{-\tau}^{0} S_{ijk}(s) \, ds \overline{\omega_k}.
\]

Comparisons of this formula with channel flow data suggests that the terms which have been truncated from equation (11) represent a relatively minor, though not necessarily negligible, effect.

A more suggestive form of equation (12) may be developed by introducing Lagrangian integral scales \( T \) and \( Q \) defined by

\[
\int_{-\tau}^{0} R_{ijk}(s) \, ds \equiv T \overline{u_i u_k}
\]

and

\[
\int_{-\tau}^{0} S_{ijk}(s) \, ds \equiv Q \overline{u_i u_j k}.
\]

In this case equation (12) becomes

\[
\overline{u_i \omega_j} = - T \overline{u_i u_k} \frac{\partial \overline{\omega_j}}{\partial x_k} + Q \overline{u_i u_j k} \overline{\omega_k},
\]

where it should be emphasized that \( T \) and \( Q \) are introduced mainly for notational convenience. In fact, there is much reason to believe that the Lagrangian scales corresponding to different choices of the indices in equations (13) and (14) will be unequal. Thus, technically, \( T \) and \( Q \) should actually carry some indication of this dependence. Such generality is allowed in the case of channel flow treated in the next section, though, for simplicity, it is avoided here and in the further discussion of the general case given in Section 5.

A second point to make is that the definitions of \( T \) and \( Q \) in equations (13) and (14) may or may not be permissible at points where a component of \( u_i u_j \) or \( \overline{u_i u_j k} \) vanishes. This depends on whether or not the left-hand side of the particular equation is zero at the same point also. For the purposes of this study, however, it is assumed in these cases, even if the relation is not exactly true, that nonetheless the magnitude of the relevant integral in equations (13) or (14) is sufficiently small so that approximation by zero is a reasonable step to take. As it turns out, this assumption does not appear to have any adverse consequences in the treatment of channel flow.

It may be noticed that in equation (15) the vorticity fluxes on the left-hand side of the equation are composed of correlations of the type \( \overline{u_i u_j k} \) which appear on the right-hand side as well. Due to this apparent circularity, it is evident that the transport law in this form is not useful in a computational sense. However, by an algebraic manipulation which does not involve the introduction of any new assumptions, a computationally viable form of equation (15) can be derived. Before considering this extension of the basic result for the general case, however, it is enlightening first to examine the form and validity of the transport law for a channel flow where direct numerical simulation data is available with which to compare its predictions with the computed correlations.
3. Application to Channel Flow

The nine specific predictions of the previous theory concerning the components of vorticity transport in a channel flow are now examined. For this discussion the notation \((x, y, z)\) for the streamwise, wall-normal, and spanwise coordinates is adopted, while \((u, v, w)\) denotes the corresponding velocities. \(\bar{\Omega} = \Omega_3 = -\bar{U}_y\) is used to indicate the mean spanwise vorticity component, and, where convenient, the notation \((\_)_x = d(\_)/dx\) is utilized. In the following, considerable use is made of the relations \(u\bar{w} = \bar{u}w\), \((\bar{u}w)_x = \bar{u}_xw + \bar{w}_xu\), \(\bar{w}w_x = \frac{1}{2}(\bar{w}^2)_x\), and so on, which apply to a channel flow.

Consider first the three helicity type fluxes \(u\omega_i\), \(i = 1, 2, 3\). It follows from equation (15) that
\[ u\omega_1 = Q\bar{u}\bar{w}_y\bar{\Omega} = 0, \]
\[ u\omega_2 = Q\bar{v}\bar{w}_z\bar{\Omega} = 0, \]
\[ u\omega_3 = Q\bar{w}\bar{w}_x\bar{\Omega} = 0, \]
so that, in effect, each of these fluxes is predicted to be zero. However, by considering the requirement of symmetry with respect to reflections in the \(x\)-\(y\) plane, it may be shown that the three helicity correlations must vanish identically. Thus, it is apparent from equations (16)–(18) that the truncation involved in deriving equation (15) is fully consistent with this condition.

Substituting the vorticity definitions
\[ \omega_1 = w_y - v_z, \]
\[ \omega_2 = u_z - w_x, \]
\[ \omega_3 = v_x - u_y, \]
into equations (16)–(18) and employing several identities appropriate to channel flow gives
\[ \bar{u}\bar{w}_y = \bar{w}_z = -\bar{w}_x = -\bar{w}_x = -\bar{u}_x = -\bar{u}_x. \]

The equality of the first and last terms in this relation implies that each of the indicated correlations are zero. As a consequence of this result and equation (15) it follows that
\[ u\omega_2 = Q\bar{u}\bar{w}_y\bar{\Omega} = 0 \]
and
\[ u\omega_1 = Q\bar{u}\bar{w}_y\bar{\Omega} = 0. \]

In other words, the streamwise flux of wall-normal vorticity and the wall-normal flux of streamwise vorticity are both predicted to be zero. As in the case of equations (16)–(18), a symmetry argument may be used to show that \(\bar{u}\omega_2\) and \(\bar{v}\omega_1\) must be zero in channel flow. Thus, once again, the transport law as given in equation (15) is able to fulfill identically the requirements of symmetry.

The four remaining transport correlations according to equation (15) take the form
\[ \bar{u}\omega_3 = -T_2\bar{w}\frac{d\bar{\Omega}}{dy} + Q_1\bar{u}\bar{w}_2\bar{\Omega}, \]
\[ \bar{v}\omega_3 = -T_1\bar{v}\frac{d\bar{\Omega}}{dy} + Q_2\bar{v}\bar{w}_3\bar{\Omega}, \]
\[ \bar{w}\omega_1 = Q_3\bar{w}\bar{u}_1\bar{\Omega}, \]
\[ \bar{w}\omega_2 = Q_4\bar{w}\bar{u}_2\bar{\Omega}, \]
where the following particular time-scale definitions have been made:
\[ T_1 = \int_{-\infty}^{0} R_{113}(s) \, ds, \]
\[ T_2 = \int_{-\infty}^{0} R_{123}(s) \, ds, \]
\[ Q_1 = \int_{-\infty}^{0} S_{113}(s) \, ds, \]
Here, normalized correlation functions such as \( R^*_k(s) \equiv R_{ik}(s)/R_{ik}(0) \) are being used. Gradient transport is seen to make a contribution only to the streamwise and wall-normal fluxes of spanwise vorticity represented in equations (21) and (22), respectively. First-order nongradient terms appear for all of the transport components. In the case of the spanwise fluxes of streamwise and wall-normal vorticity given in equations (23) and (24), respectively, this is the sole source of correlation.

From direct numerical-simulation data it is possible to make tests of the predicted vorticity fluxes given in equations (21)–(24). In principle, this can include evaluation of all the quantities appearing in these relations, including the time scales. As a practical matter, however, it is technically difficult to compute the necessary time scales since this requires assembling a very large ensemble of particle paths having endpoints at a set of \( y \) values distributed across the channel. While it is hoped that one day this information can be obtained, for the moment, tests of the transport law have been performed in which only the Eulerian one-point correlations appearing in equations (21)–(24) have been computed, while the time scales have been assigned constant positive values. As will be seen below, even with this simplifying step, the predicted fluxes display a strong qualitative agreement with their values computed directly from the simulated flow field.

The data used in this study to calculate the vorticity fluxes as well as to check the transport law is from a channel flow simulation at Reynolds number \( Re_c = 250 \) based on friction velocity, \( U_t \), and channel width, \( h \), performed by R. Leighton and R. Handler at the Naval Research Laboratory, Washington, DC (Handler et al., 1989). The quality of the simulation as measured by the Reynolds stresses and other turbulence statistics is similar to that of others computed elsewhere (Kim et al., 1987). The Eulerian correlations appearing in the transport formulas were obtained from planar averages over 28 statistically independent realizations of the flow which had been stored on disk files. The constant values assigned to the scales were \( T^*_1 = 4.8, T^*_2 = 12.3, Q^*_1 = 16.3, Q^*_2 = 5.5, Q^*_3 = 0.95, \) and \( Q^*_4 = 9.5 \), where the superscript “+” signifies quantities scaled by \( U_t \) and the kinematic viscosity.

Figures 1–4 display the computed values of the fluxes \( \vec{u}_3 \), \( \vec{v}_3 \), \( \vec{w}_1 \), and \( \vec{w}_2 \) which have been predicted above to be nonzero, together with a numerical evaluation of their forms as given in equations (21)–(24), respectively. In each case, the validity of the predicted expression appears to be confirmed, particularly with regards to its variation in sign and position and the relative magnitude of

![Figure 1](image1.png)  
**Figure 1.** Vorticity flux \( \vec{u}_3 \), • computed, —— as predicted in equation (21).  

![Figure 2](image2.png)  
**Figure 2.** Vorticity flux \( \vec{v}_3 \), • computed, —— as predicted in equation (22).
its local maxima and minima. While the next section considers in detail the flow processes underlying the trends in these curves, it may be pointed out here that the clear concurrence between the predicted and computed fluxes is very much rooted in the particular forms taken by the derived nongradient transport terms. For example, in the case of both $\omega \overline{u}_1$ and $\overline{u}_1 \omega_3$ the nongradient terms are opposite in sign to the gradient term and are thus directly responsible for matching the large near-wall peaks observed in the simulation data. For the fluxes $\omega \overline{u}_2$ and $\overline{u}_2 \omega_3$, for which gradient transport does not contribute, the predicted nongradient terms behave almost exactly as needed to account for the trends in the simulation data.

The discrepancies between the predicted and computed fluxes which are visible in Figures 1–4 can be readily attributed to likely spatial variations in the time scales, which are not taken into account, as well as, possibly, the exclusion from equation (15) of the terms $\Phi_1$, $\Phi_2$, and $\Phi_3$ appearing in equation (11). Some small errors may also arise from the numerical simulation data itself. That the use of constant scales introduces an error is suggested by the behavior of equation (22) near the centerline as shown in Figure 2. In particular, the nongradient term in equation (22) is identically zero at the centerline so that any error between the left- and right-hand sides of the equation at this point may be due entirely to the value assigned to $T_1$.

As discussed previously, the predicted fluxes contained in equations (21)–(24) are not yet in what may be considered to be a useful computational form since they depend on correlations such as $\overline{u}_2 \overline{w}_1$, $\overline{v}_1 \overline{w}_2$, and so on. However, without introducing additional assumptions this dependence may be eliminated. To see this, consider the following development. Replace the vorticity components appearing in the left-hand side of equations (23) and (24) with their definitions. This yields

$$\frac{1}{2} \frac{d \overline{w}_2^2}{dy} - \overline{w}_2 \overline{w}_3 = Q_3 \overline{w}_1 \overline{u}_3 \overline{v}_1 \Omega \tag{25}$$

and

$$\overline{w}_1 = Q_4 \overline{w}_2 \overline{v}_1 \overline{u}_3 \overline{u}_1 \Omega \tag{26}$$

which consist of two equations in the two unknowns $\overline{w}_1$ and $\overline{w}_2$. A calculation then gives

$$\overline{w}_2 = \frac{\frac{1}{2}(d \overline{w}_2^2/dy)}{1 + Q_3 Q_4 \Omega^2}. \tag{27}$$

This result is fundamental to the whole approach since with it each of the four nonzero vorticity fluxes can be placed into a computable form in terms of $\Omega$, the Reynolds stresses, and the time scales. In fact, a calculation using equations (26), (27), and identities such as $\overline{v}_2 = -\overline{w}_1$ gives

$$\omega \overline{w}_3 = -T_2 \frac{d \Omega}{dy} - \frac{\frac{1}{2}(d \overline{w}_2^2/dy)Q_1 Q_4 \Omega^2}{1 + Q_3 Q_4 \Omega^2}. \tag{28}$$
\[ \frac{\partial\bar{\omega}_3}{\partial y} = -T_1\bar{v}^2\frac{d\bar{\Omega}}{dy} - \frac{1}{2}(\frac{\partial \bar{w}^2}{\partial y})\frac{Q_2\bar{\Omega}}{1 + Q_3Q_4\bar{\Omega}^2}, \quad (29) \]

\[ \bar{w}\bar{\Omega}_1 = \frac{\frac{1}{2}(\frac{\partial \bar{w}^2}{\partial y})Q_2Q_4\bar{\Omega}^2}{1 + Q_3Q_4\bar{\Omega}^2}, \quad (30) \]

and

\[ \bar{w}\bar{\Omega}_2 = \frac{\frac{1}{2}(\frac{\partial \bar{w}^2}{\partial y})Q_4\bar{\Omega}}{1 + Q_3Q_4\bar{\Omega}^2}. \quad (31) \]

Keeping the same values for the scales used previously, a series of tests of equations (27)–(31) have been performed using the direct simulation data. Figure 5 shows a check of equation (27) in which the qualitative agreement is seen to be excellent. Tests of the vorticity fluxes in equations (28)–(31) are shown in Figures 6–9, respectively. The overall agreement here is quite good, particularly in view of the potential for compounding of errors which may be expected to result from the use of equation (27), which is itself not exactly satisfied, in deriving equations (28)–(31).

It is of some interest to examine the implication of the preceding result insofar as the prediction of the mean velocity of a turbulent channel flow is concerned. In particular, the transport correlations in equations (29) and (31) when substituted into the vorticity-transport form of the momentum equation, i.e.,

\[ 0 = 2 + \frac{1}{Re} \frac{d^2\bar{U}}{dy^2} + \bar{w}\bar{\Omega}_3 - \bar{w}\bar{\Omega}_2, \]

yields

\[ 0 = 2 + \left( \frac{1}{Re} + T_1\bar{v}^2 \right)\frac{d^2\bar{U}}{dy^2} + \frac{1}{2} \frac{d\bar{U}}{dy} \frac{\partial \bar{w}^2}{\partial y} + \frac{Q_2 + Q_4}{1 + Q_3Q_4\bar{\Omega}^2}, \quad (32) \]

which, in effect, is the closed form of the momentum equation predicted by the present theory. In this relation variables have been scaled using \( U \), \( \bar{v} \), and \( h \). The last term in equation (32) represents the contribution of the nongradient vorticity fluxes. In view of the previous calculations, this term can be expected to be of some significance.

To examine the usefulness of equation (32) as a closure to the momentum equation, a series of calculations were performed in which it was solved numerically for \( \bar{U} \). In this the channel flow simulation data of Kim et al. (1987) at \( Re = 360 \) was used to supply values of \( \bar{v}^2 \) and \( \bar{w}^2 \). A typical result of this computation is shown in Figure 10, where it may be seen that the agreement with the data is quite good. For this calculation constant scale values \( Q_2^* = 7.92 \), \( Q_3^* = 0.65 \), and \( Q_4^* = 10.8 \) were used. To allow for greater accuracy, \( T_1^* \) was increased smoothly toward the centerline in conformity to the observation made previously with regard to Figure 2. For the curve in Figure 10, \( T_1^* = 2.0 \) at the wall and \( T_1^* = 25.2 \) at the centerline.

Thus far, none of the previous results have made explicit use of the continuity equation. However, this relation has several implications for the theory which are worth exploring. In particular, by multiplying the continuity equation \( \bar{u}_x + \bar{v}_y + \bar{w}_z = 0 \) in turn by \( \bar{u}, \bar{v}, \) and \( \bar{w} \) and then averaging, the following relationships result:

\[ \bar{u}\bar{w}_x + \bar{w}\bar{v}_y + \bar{w}\bar{w}_z = 0, \quad (33) \]

\[ \bar{w}\bar{u}_x + \bar{w}\bar{v}_y + \bar{w}\bar{w}_z = 0, \quad (34) \]

and

\[ \bar{w}\bar{u}_x + \bar{w}\bar{v}_y + \bar{w}\bar{w}_z = 0. \quad (35) \]

It may be inferred from equations (19) and (20) that equation (35) is identically zero in a channel flow. Equation (33), after using equations (29) and (31), yields

\[ \frac{d\bar{w}}{dy} = T_1\bar{v}^2\frac{d\bar{\Omega}}{dy} + \frac{1}{2} \frac{d\bar{w}^2}{dy} \frac{(Q_2 + Q_4)\bar{\Omega}}{1 + Q_3Q_4\bar{\Omega}^2}. \quad (36) \]

The primary interest of this relation lies in the fact it can be used to extract the Reynolds stress
Turbulent Vorticity Transport in Three Dimensions

Figure 5. Correlation $\Omega_{ij}$, • computed, --- as predicted in equation (27).

Figure 6. Vorticity flux $\Omega_{ij}$, • computed, --- as predicted in equation (28).

Figure 7. Vorticity flux $\Omega_{ij}$, • computed, --- as predicted in equation (29).

Figure 8. Vorticity flux $\Omega_{ij}$, • computed, --- as predicted in equation (30).

Figure 9. Vorticity flux $\Omega_{ij}$, • computed, --- as predicted in equation (31).

Figure 10. Mean velocity field, • computed by Kim et al. (1987), --- computed from equation (32).
closure which is implicit in equation (32). Thus, integrating equation (36) from 0 → y gives
\[ \overline{uv} = -T_1 \overline{v^2} \frac{d\overline{U}}{dy} + \int_0^y \frac{d\overline{U}}{dy} \left( \frac{d(T_1 \overline{v^2})}{dy} - \frac{1}{2} \frac{(Q_2 + Q_4)}{1 + Q_2 \overline{\Omega}^2} \frac{d\overline{w^2}}{dy} \right) dy \]

in which the Reynolds stress is modeled as the sum of a gradient term plus a nonlocal integral expression accounting for nongradient transport effects.

The meaning of equation (37) can be better understood by comparing it to the momentum transport law resulting from a direct treatment of \( \overline{uv} \) by a Lagrangian analysis similar to that which led to equation (15). In this case one substitutes for \( u \) in \( \overline{uv} \) an expression derived from integrating the x-momentum equation along a particle path. For \( \tau \) large enough the result is (Bernard et al., 1989a, b)
\[ \overline{uv} = -T_1 \overline{v^2} \frac{d\overline{U}}{dy} - \int_{t_0}^{t_0 + \tau} \frac{v(t_0 + s)}{\partial x} ds + O(\tau^2) + O \left( \frac{\tau}{Re} \right). \]

Comparison of equations (37) and (38) suggests that the integral terms in these relations are equivalent. In equation (38) the expression containing the integral represents the cumulative action of the pressure force in modifying the momentum of fluid particles along their paths. Consequently, a similar interpretation may be applied to the integral term in equation (37). The relationship between these terms raises many interesting questions which it is intended to examine in future studies.

In Figure 11 the Reynolds stress computed in the simulation at \( Re_\tau = 250 \) is plotted together with its gradient transport component as given in equation (37). The difference between these curves represents the nongradient transport contribution. It is clear from this that a gradient model of the Reynolds stress is a good representation only near the wall, in agreement with an earlier conclusion of Taylor (1935, 1937).

Consider now the last of the three identities derived from the continuity equation, specifically equation (34). Applying equations (28) and (30) to this yields the following interesting relation:
\[ \frac{1}{2} \frac{d(\overline{v^2} - \overline{\tilde{u}^2})}{dy} = -T_2 \overline{uv} \frac{d\tilde{\Omega}}{dy} + \frac{1}{2} \frac{d\overline{w^2}}{dy} 1 - Q_2 \tilde{\Omega}^2 \frac{d\overline{w^2}}{dy} \]

Since \( \overline{uv} \) can be calculated directly from \( \tilde{\Omega} \) using the well-known channel flow identity
\[ \overline{uv} = 2y - 1 - \frac{\tilde{\Omega}}{Re}, \]
equation (39) expresses a condition between the three normal Reynolds stresses, \( \tilde{\Omega} \), and the time scales. Numerical evaluation of (39) shows it to be satisfied to the same degree of accuracy as were equations (28)–(31). The role that this equation might play in the prediction of turbulent flows will be an object of future study.
It is of some interest to interpret the preceding results concerning the fluxes of vorticity in light of the vortical model of the turbulent wall region which has emerged in recent years (Robinson et al., 1988). In this, "horseshoe" vortices or, perhaps, more often, pieces of such structures, play a major role in the dynamics of the wall region. In particular, they are likely to have an influence upon the vorticity fluxes adjacent to a fixed boundary.

A close examination of the nongradient contributions to equations (21)–(24) reveals that they are formed from the mean product of a velocity component, u, v, or w with one of the quantities $\omega_1 = \overline{\Omega_1} \int u_s \, ds$, $\omega_2 = \overline{\Omega_1} \int v_z \, ds$, or $\omega_3 = \overline{\Omega_1} \int w_z \, ds$. The factor $\omega_1$ represents the instantaneous contribution to $\omega_2$ resulting from shearing of $\overline{\Omega}$ into the streamwise direction. Similarly, $\omega_2$ arises from shearing of $\overline{\Omega}$ into the wall normal direction and $\omega_3$ accounts for that part of $\omega_3$ created by stretching of $\overline{\Omega}$ along its axis. Keeping these definitions in mind, it is possible to interpret the trends in the vorticity fluxes shown in Figures 1–4 as they relate to the coherent structures in the wall region.

The separate gradient and nongradient contributions to $\overline{\omega_3}$ indicated in equation (22) are shown plotted in Figure 12 for the data at $Re = 250$. The gradient term is seen to be always negative in the lower half channel and contributes to a flux of $\overline{\Omega}$, which is itself negative, in the direction away from the boundary. In contrast, the nongradient term makes a significant positive contribution to transport in the region extending from the wall out to $y^+ \approx 30$. As suggested in the above discussion, the behavior of the nongradient term ultimately reflects the connection between $v$ and $\omega_3$. The region where the nongradient term is most significant, in fact, contains the fluid motions beneath the centers of streamwise vortex pairs, i.e., the legs of horseshoe vortices. The large positive magnitude of the nongradient correlation may be explained as due to the circumstance that contractive motions, where $\omega_3 > 0$, occur when $v > 0$ in the region adjacent to the wall between counterrotating vortices, and expansive motions, where $\omega_3 < 0$, occur when $v < 0$ outside the vortex pairs. Further from the wall, above the vortex centers, this process should reverse itself and the sign of the nongradient term should change, as indeed it does. The greater disorganization in the flow away from the wall can explain the evident drop off in its magnitude. It is interesting to note that according to Figures 2 and 12 the total turbulent flux of $\overline{\Omega}$ is actually countergradient for $y^+ < 10$. Clearly, a gradient vorticity transport model is inappropriate in this region.

In the case of the transport of spanwise vorticity in the streamwise direction given in equation (21), the gradient term which is plotted in Figure 13, is always positive since it has a negative coefficient of diffusion. It represents an apparent upstream turbulent flux of $\overline{\Omega}$ caused by the fact that fluid parcels tending to travel slower than the mean ($u < 0$) also tend to travel away from the wall ($v > 0$). Since the mean vorticity increases in intensity toward the wall, vorticity greater than the mean in magni-
tude, in effect, diffuses upstream. Conversely, in-rushes of fluid ($u > 0, v < 0$) bring vorticity less than the mean in magnitude downstream. The origin of the nongradient term in equation (21) lies in the coupling between $u$ and $\omega_2$. According to Figure 13, it attains significant negative values in the region closest to the wall. This corresponds to the idea that in the low-speed streak region between vortex pairs, where $u < 0$, contractive motions occur in which $\omega_2 > 0$. Outside the streaks, where $u > 0$, stretching occurs in which $\omega_2 < 0$. Figures 1 and 13 show that the nongradient term is dominant to $y^+ \approx 12$ while the gradient term is greatest beyond this point. Consequently, near the wall the net flux of spanwise vorticity is downstream while only further from the boundary is it upstream.

The correlation $w \omega_2$ in equation (24) accounts for the transport of the normal component of vorticity in the spanwise direction. It is entirely a result of any relationship that $w$ might have with $\omega_2$. According to Figure 4, this flux is negative near the wall changing to positive values at $y^+ \approx 45$. This behavior appears to be consistent with the previously described vortex structure. In particular, near the wall, below the vortex pairs, the spanwise flow is toward the central region between the legs. Where $w < 0$ the vorticity vector belonging to the leg of a lifting up spanwise mean vortex is pointed away from the wall, i.e., $\omega_2 > 0$ and on the side of the vortex pair where $w > 0$ the vorticity vector is directed toward the wall and thus $\omega_2 < 0$. This implies that $w \omega_2$ has a negative sign near the wall in agreement with Figure 4. The opposite may be assumed to happen above streamwise vortex pairs where the spanwise flow is outward from the center of the vortices. Here, the correlation is weaker presumably due to a lower degree of coherence in the flow structures.

A similar discussion applies to $w \omega_1$ accounting for the transport of streamwise vorticity in the spanwise direction given in equation (23) and plotted in Figure 3. In this case the flux, which depends on the correlation of $w$ with $\omega_1$, is seen to attain a large positive peak very close to the wall at $y^+ = 3$. Its trend may be explained by considering the perturbation induced in an initially spanwise mean vortex by a low-speed streak adjacent to the legs. In the horseshoe configuration which evolves from the disturbance, the leg for which $\omega_1 > 0$ moves in the $+z$ direction ($w > 0$) and the opposite leg moves in the $-z$ direction ($w < 0$). In this scenario, the head of the horseshoe vortex must point upstream. In its later development, as it lifts up from the wall, the vortex loop presumably becomes reoriented downstream due to the action of the mean shear flow.

5. The Transport Law for Three-Dimensional Mean Flows

In the previously considered case of channel flow, the nonzero components of vorticity transport given in equations (21)–(24) were converted to the more useful form in equations (28)–(31) without the use of additional assumptions. A similar step must be taken for the general case if equation (15) is to be of more than just academic interest. The object, then, is to replace the dependence of equation (15) on the correlations $u_i u_j$ with what may be termed computable expressions involving the Reynolds stresses, the mean vorticities, and the time scales. As mentioned previously, to simplify the forthcoming analysis, individual components of the $T$ and $Q$ scales are not introduced. In addition, it is convenient to confine the discussion to the case of rectangular cartesian coordinates.

The identity
\[
\frac{1}{2} \frac{\partial u_i^2}{\partial x_k} = \frac{1}{2} \frac{\partial u_i^2}{\partial x_k} + \frac{1}{2} \frac{\partial u_i^2}{\partial x_k} \frac{\partial \vec{\Omega}_i}{\partial x_k} \quad (i = 1, 2, 3)
\]

which holds for each fixed pair of values of the indices $i$ and $k$, implies that the three helicity type correlations in equation (15) are given by
\[
\vec{u}_i \omega_i = -T u_i u_k \frac{\partial \vec{\Omega}_i}{\partial x_k} + \frac{1}{2} \frac{\partial u_i^2}{\partial x_k} Q \Omega_k \quad (i = 1, 2, 3)
\]

so that they need be analyzed no further.

Before considering the remaining fluxes, it is helpful to note the following consequence of equation (42). From the definition of the vorticity it follows that
\[
\vec{u}_1 \omega_1 = \vec{u}_1 u_{3,2} - \vec{u}_1 u_{2,3},
\]
\[
\vec{u}_2 \omega_2 = \vec{u}_2 u_{1,3} - \vec{u}_2 u_{3,1},
\]
The correlations \( \bar{u}_1 u_{2,3} \), \( \bar{u}_2 u_{3,1} \), and \( \bar{u}_3 u_{1,2} \) in equations (43)–(45) may be replaced using the identities

\[
\bar{u}_1 u_{2,3} + \bar{u}_2 u_{1,3} = (\bar{u}_1 u_2)_i, \quad i = 1, 2, 3, \quad (46)
\]
\[
\bar{u}_2 u_{3,1} + \bar{u}_3 u_{2,1} = (\bar{u}_2 u_3)_i, \quad i = 1, 2, 3, \quad (47)
\]
and

\[
\bar{u}_3 u_{1,2} + \bar{u}_1 u_{3,2} = (\bar{u}_3 u_1)_i, \quad i = 1, 2, 3, \quad (48)
\]
to yield the coupled set of equations

\[
\bar{u}_1 \bar{u}_3 + \bar{u}_3 \bar{u}_1 = \bar{u}_1 \bar{u}_3 + (\bar{u}_1 u_2)_3,
\]
\[
\bar{u}_2 \bar{u}_3 + \bar{u}_3 \bar{u}_2 = \bar{u}_2 \bar{u}_3 + (\bar{u}_2 u_3)_1,
\]
\[
\bar{u}_3 \bar{u}_1 + \bar{u}_1 \bar{u}_3 = \bar{u}_3 \bar{u}_1 + (\bar{u}_3 u_1)_2,
\]

for the unknowns \( \bar{u}_1 \bar{u}_3 \), \( \bar{u}_3 \bar{u}_2 \), and \( \bar{u}_2 \bar{u}_1 \). Solving these gives

\[
\bar{u}_1 \bar{u}_3 = \frac{1}{2}(\bar{u}_1 \bar{u}_3 + \bar{u}_3 \bar{u}_1) + \frac{1}{2}(\bar{u}_1 \bar{u}_3 + \bar{u}_3 \bar{u}_1),
\]
\[
\bar{u}_2 \bar{u}_3 = \frac{1}{2}(\bar{u}_2 \bar{u}_3 + \bar{u}_3 \bar{u}_2) + \frac{1}{2}(\bar{u}_2 \bar{u}_3 + \bar{u}_3 \bar{u}_2),
\]
\[
\bar{u}_3 \bar{u}_1 = \frac{1}{2}(\bar{u}_3 \bar{u}_1 + \bar{u}_1 \bar{u}_3) + \frac{1}{2}(\bar{u}_3 \bar{u}_1 + \bar{u}_1 \bar{u}_3),
\]

These relations, when used with equations (42) and (46)–(48) express the six correlations \( \bar{u}_i \bar{u}_j \), \( i \neq j \), in terms of the Reynolds stresses, mean vorticities, and time scales.

According to equation (15), the six off-diagonal vorticity transport correlations are given by

\[
\bar{u}_1 \bar{u}_3 = \frac{\partial \bar{u}_3}{\partial x_k} - \frac{\partial \bar{u}_1}{\partial x_k}, \quad (52)
\]
\[
\bar{u}_2 \bar{u}_1 = \frac{\partial \bar{u}_1}{\partial x_k} - \frac{\partial \bar{u}_2}{\partial x_k}, \quad (53)
\]
\[
\bar{u}_3 \bar{u}_2 = \frac{\partial \bar{u}_2}{\partial x_k} - \frac{\partial \bar{u}_3}{\partial x_k}, \quad (54)
\]
\[
\bar{u}_3 \bar{u}_1 = \frac{\partial \bar{u}_1}{\partial x_k} - \frac{\partial \bar{u}_3}{\partial x_k}, \quad (55)
\]
\[
\bar{u}_2 \bar{u}_3 = \frac{\partial \bar{u}_3}{\partial x_k} - \frac{\partial \bar{u}_2}{\partial x_k}, \quad (56)
\]
\[
\bar{u}_1 \bar{u}_2 = \frac{\partial \bar{u}_2}{\partial x_k} - \frac{\partial \bar{u}_1}{\partial x_k}, \quad (57)
\]

Rearranging and utilizing equations (46)–(48) the following coupled system of equations for the unknowns \( \bar{u}_1 \bar{u}_2, \bar{u}_2 \bar{u}_3, \bar{u}_3 \bar{u}_1 \), \( \bar{u}_1 \bar{u}_3 \), and \( \bar{u}_2 \bar{u}_1 \) is obtained:

\[
\bar{u}_1 \bar{u}_2 = Q\bar{u}_3 + Q\bar{u}_1, \quad f_1 = \bar{u}_1 \bar{u}_2, \quad (58)
\]
\[
\bar{u}_2 \bar{u}_3 = Q\bar{u}_1 + Q\bar{u}_2, \quad f_2 = \bar{u}_2 \bar{u}_3, \quad (59)
\]
\[
\bar{u}_3 \bar{u}_1 = Q\bar{u}_2 + Q\bar{u}_3, \quad f_3 = \bar{u}_3 \bar{u}_1, \quad (60)
\]
\[
\bar{u}_1 \bar{u}_3 = Q\bar{u}_1 + Q\bar{u}_3, \quad f_4 = \bar{u}_1 \bar{u}_3, \quad (61)
\]
\[
\bar{u}_2 \bar{u}_1 = Q\bar{u}_2 + Q\bar{u}_1, \quad f_5 = \bar{u}_2 \bar{u}_1, \quad (62)
\]
\[
\bar{u}_3 \bar{u}_2 = Q\bar{u}_3 + Q\bar{u}_2, \quad f_6 = \bar{u}_3 \bar{u}_2, \quad (63)
\]
where

\[ f_1 = \frac{1}{2} (u_1^2),_2 - Tu_1 u_k \frac{\partial \Omega_3}{\partial x_k} + (u_3 u_1)_3 Q \Omega_3 + Q \Omega_2 u_1 u_3, \]

\[ f_2 = \frac{1}{2} (u_2^2),_3 - Tu_2 u_k \frac{\partial \Omega_1}{\partial x_k} + (u_1 u_2)_1 Q \Omega_1 + Q \Omega_3 u_2 u_1, \]

\[ f_3 = \frac{1}{2} (u_3^2),_1 - Tu_3 u_k \frac{\partial \Omega_2}{\partial x_k} + (u_2 u_3)_2 Q \Omega_2 + Q \Omega_3 u_3 u_1, \]

\[ f_4 = \frac{1}{2} (u_2^2),_2 + Tu_3 u_k \frac{\partial \Omega_1}{\partial x_k} - (u_3 u_1)_1 Q \Omega_1 - Q \Omega_3 u_3 u_1, \]

\[ f_5 = \frac{1}{2} (u_1^2),_3 + Tu_1 u_k \frac{\partial \Omega_2}{\partial x_k} - (u_1 u_2)_2 Q \Omega_2 - Q \Omega_3 u_1 u_3, \]

\[ f_6 = \frac{1}{2} (u_3^2),_1 + Tu_2 u_k \frac{\partial \Omega_3}{\partial x_k} - (u_2 u_3)_3 Q \Omega_3 - Q \Omega_3 u_2 u_3. \]

Equations (58)-(63) may be written as a block 2 x 2 matrix system for the vectors \((u_1 u_2, u_2 u_3, u_3 u_1, u_2 u_3, u_3 u_1, u_3 u_1, u_2 u_3, u_3 u_1, u_2 u_3)\). Inversion of this equation results in two separate 3 x 3 matrix problems for each of the vectors. The solution of these gives

\[ \bar{u}_1 u_2 = f_1 + J_1 / D, \]

\[ \bar{u}_2 u_3 = f_2 + J_2 / D, \]

\[ \bar{u}_3 u_1 = f_3 + J_3 / D, \]

\[ \bar{u}_2 u_3 = f_4 + J_1 / D, \]

\[ \bar{u}_3 u_1 = f_5 + J_2 / D, \]

\[ \bar{u}_3 u_1 = f_6 + J_3 / D, \]

where

\[ D = 1 + Q^2 (\Omega_1^2 + \Omega_2^2 + \Omega_3^2), \]

\[ J_1 = K_1 (1 + Q^2 \Omega_3^2) + K_2 (Q \Omega_1 + Q^2 \Omega_2 \Omega_3) - K_3 (Q \Omega_3 - Q^2 \Omega_1 \Omega_2), \]

\[ J_2 = K_2 (1 + Q^2 \Omega_3^2) + K_3 (Q \Omega_2 + Q^2 \Omega_3 \Omega_1) - K_1 (Q \Omega_1 - Q^2 \Omega_3 \Omega_2), \]

\[ J_3 = K_3 (1 + Q^2 \Omega_3^2) + K_1 (Q \Omega_3 + Q^2 \Omega_1 \Omega_2) - K_2 (Q \Omega_1 - Q^2 \Omega_3 \Omega_2), \]

\[ K_1 = Q \Omega_1 f_3 - Q \Omega_3 f_3, \]

\[ K_2 = Q \Omega_2 f_6 - Q \Omega_1 f_3, \]

\[ K_3 = Q \Omega_3 f_4 - Q \Omega_2 f_2, \]

Finally, substituting equations (64)-(69) into equations (52)-(57) gives

\[ \bar{u}_1 \omega_3 = -Tu_1 u_k \frac{\partial \Omega_3}{\partial x_k} + \frac{f_1}{D} + Q \Omega_3 (u_3 u_1)_3 + Q \Omega_2 u_1 u_3, \]

\[ \bar{u}_2 \omega_1 = -Tu_2 u_k \frac{\partial \Omega_1}{\partial x_k} + \frac{f_2}{D} + Q \Omega_1 (u_1 u_2)_1 + Q \Omega_3 u_2 u_1, \]

\[ \bar{u}_3 \omega_2 = -Tu_3 u_k \frac{\partial \Omega_2}{\partial x_k} + \frac{f_3}{D} + Q \Omega_2 (u_2 u_3)_2 + Q \Omega_1 u_3 u_1, \]

\[ \bar{u}_3 \omega_1 = -Tu_3 u_k \frac{\partial \Omega_1}{\partial x_k} - \frac{f_1}{D} + Q \Omega_1 (u_3 u_1)_1 + Q \Omega_2 u_3 u_1, \]

\[ \bar{u}_1 \omega_2 = -Tu_1 u_k \frac{\partial \Omega_2}{\partial x_k} - \frac{f_2}{D} + Q \Omega_2 (u_1 u_2)_2 + Q \Omega_3 u_1 u_2, \]

\[ \bar{u}_2 \omega_3 = -Tu_2 u_k \frac{\partial \Omega_3}{\partial x_k} - \frac{f_3}{D} + Q \Omega_3 (u_2 u_3)_3 + Q \Omega_1 u_2 u_3. \]
where the last correlation in each of these equations can be computed using equations (49)-(51).

Examination of the definitions of $J_1 \rightarrow J_3$ and $f_1 \rightarrow f_6$ appearing in equations (70)-(75) shows that these quantities are functions only of $u_i u_j$, $\Omega$, and the time scales. Consequently, equations (70)-(75) have this dependence as well, so that the goal of expressing the vorticity fluxes in a computationally useful form has been met. The essence of this development has been to use the 27 equations contained in (41), (42), (46)-(48), and (52)-(57) to express the 27 components of $\bar{u}_i u_j$ in terms of the chosen unknowns.

Up to this point the Reynolds shear stresses have been included in the development of the revised transport law given in equations (70)-(75). This dependence can be eliminated, in principle, using the three equations (33)-(35) which were derived previously from the continuity condition. For a general three-dimensional mean flow these relations yield

$$
(u_1 u_2)_2 + (u_3 u_1)_3 = f_3 + f_6 + 2J_3/D - \frac{1}{2}(u_2^2)_1,
$$

$$
(u_2 u_3)_3 + (u_1 u_2)_1 = f_1 + f_4 + 2J_4/D - \frac{1}{2}(u_2^2)_2,
$$

$$
(u_3 u_1)_1 + (u_2 u_3)_2 = f_2 + f_5 + 2J_2/D - \frac{1}{2}(u_2^2)_3,
$$

in which use has been made of equations (70)-(75). Equations (76)-(78) are three coupled differential equations which can be used in the determination of the Reynolds shear stresses $u_i u_j$, $u_3 u_3$, and $u_i u_3$.

The present theory culminates in a three-dimensional vorticity-transport law given by equations (42) and (70)-(75). Beyond any intrinsic interest this result may have as an expression of the physics of turbulent transport, it may also be taken as the first step in the development of a turbulence closure scheme. In this, equations (70)-(75) are substituted into the vorticity transport form of the averaged momentum equation. The resulting relations must be supplemented by the additional equations (76)-(78) in order to eliminate the off-diagonal components of the Reynolds stress tensor. It is evident that, by allowing for the generality of three-dimensional flows, a very large increase in the complexity of vorticity transport theory has resulted. Comparison of the formulas of this section with those of the previous case of channel flow underscores this difference. Only further research will be able to assess the practical side of the general form of the transport law. However, it may be observed that, for the large class of turbulent flows whose mean properties vary only in two dimensions, the complexity of the present result is diminished to the point where there does not seem to be a question as to the potential usefulness of the theory. In particular, for such flows it may be shown that each of the correlations in equations (42) and (49)-(51) are identically zero. This taken with the fact that $\Omega_1 = \Omega_2 = 0$ brings equations (70)-(75) to a relatively modest scale.

6. Conclusions

A three-dimensional turbulent vorticity-transport law has been derived in which nongradient terms representing the effects of vorticity stretching and reorientation on transport were included in a useful form. The approach rests on a Lagrangian analysis which, while having much in common with the previous vorticity-transport theory of G.I. Taylor, avoids the limitations of that approach by incorporating alternative Lagrangian integral expressions. The predictions of the theory for the case of channel flow were tested using data derived from direct numerical simulations. The five, of nine, vorticity fluxes which were indicated by the theory to be zero in channel flow appear to satisfy this condition. For the remaining four fluxes, the theory was able to account for the greatest share of their computed trends. Closer agreement between the theory and the simulated data appeared to be attainable if some allowance were made for variation in the Lagrangian time scales, as well as, possibly, higher-order effects which were not included in the derived transport law.

With a view toward developing a complete and self-contained turbulence closure based on the present results, a test of the theory in the capacity of a closure to the momentum equation was carried out. This revealed that good predictions of the mean velocity field could be attained with relatively minor allowances for spatial variation in the time scales. Nongradient effects in the transport law
made an essential contribution to these results. An attempt was made to use the transport law to provide a physical explanation for the trends of the vorticity fluxes which were observed in the direct simulation data. By analyzing the sources of correlation in the nongradient terms in the transport law, a plausible connection was established between the values calculated in the simulations and the values which would be expected to occur from coherent vortical events in the wall region.

A momentum-transport law for channel flow was derived as a further consequence of the vorticity-transport law. This consisted of the sum of a gradient term and an integral expression accounting for the effect of the pressure field in modifying the momentum of fluid particles. Numerical evaluation of the terms showed that the gradient term by itself could be considered a good representation of momentum transport only near the boundary. In the case of vorticity transport the opposite was found to be true, namely, that a gradient model could be justified only at points at a distance from fixed walls. Both of these results agree with much earlier findings of Taylor.

This study suggests a number of topics which it may be useful to examine in the future. In particular, it would be of interest to assemble an ensemble of Lagrangian particle path data with which the validity of equation (15) could be tested at a fundamental level. Such a study would more precisely reveal the importance of the higher-order effects which have been truncated in the current work. Detailed Lagrangian calculations can also produce values of the time scales which would be of great benefit in discovering how the transport law might be used to advantage in diverse applications. An immediate goal in future work is to initiate a study into the effectiveness of equations (70)–(75) in describing the class of flows such as boundary layers whose mean fields vary only in two dimensions.

References


