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Kinematics of velocity and vorticity correlations in turbulent flow

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The kinematic problem of calculating second-order velocity moments from given values of the vorticity covariance is examined. Integral representation formulas for second-order velocity moments in terms of the two-point vorticity correlation tensor are derived. The special relationships existing between velocity moments in isotropic turbulence are expressed in terms of the integral formulas yielding several kinematic constraints on the two-point vorticity correlation tensor in isotropic turbulence. Numerical evaluation of these constraints suggests that a Gaussian curve may be the only form of the longitudinal velocity correlation coefficient which is consistent with the requirement of isotropy. It is shown that if this is the case, then a family of exact solutions to the decay of isotropic turbulence may be obtained which contains Batchelor’s final period solution as a special case. In addition, the computed results suggest a method of approximating the integral representation formulas in general turbulent shear flows.

I. INTRODUCTION

In second-order vorticity-based turbulence closures such as the mean vorticity and covariance (MVC) closure, the kinematic problem arises of computing, at a fixed time, the velocity covariance \( \langle u_i u_j \rangle \) from given values of the vorticity covariance \( \langle w_i w_j \rangle \). Here \( u_i \) and \( w_i \) are the velocity and vorticity fluctuation vectors, and \( \langle \cdot \rangle \) denotes the ensemble average of \( \cdot \). The velocity covariance \( \langle u_i u_j \rangle \) is needed both to compare with experimental values and to be used in modeling unclosed terms in the averaged vorticity and vorticity covariance equations. Some of the closed terms in the MVC closure also depend on the correlations \( \langle u_i u_j \rangle \) and \( \langle u_{im} u_{jn} \rangle \), where \( u_{ik} = \partial u_i / \partial x_k \), and, consequently, these must also be computed from \( \langle w_i w_j \rangle \).

von Karman and Howarth\(^5\) derived integral representation formulas for certain velocity derivative-vorticity correlations, by averaging the product of the vorticity field and derivatives of the integral representation of the velocity field in terms of the vorticity field. The resulting formulas depend on the spatial distribution of the two-point vorticity correlation tensor, \( \langle w_i w_j \rangle = \langle w_i (x) w_j (x + r) \rangle \). More recently, Chorin\(^4,5\) developed equations giving \( \langle u_i u_j \rangle \) in terms of \( \langle w_i w_j \rangle \). This was done by applying a coarse graining hypothesis (to the effect that the circulations of disjoint regions in a turbulent flow are independent Gaussian random variables) to the average of the product of the integral representations of the velocity field. Subsequently, Bernard\(^1\) and Bernard and Berger\(^2\) showed that if the same derivation is carried out without imposing simplifying assumptions (such as the coarse graining hypothesis), then the resulting representation formula for \( \langle u_i u_j \rangle \) depends on \( \langle w_i w_j \rangle \) and not just \( \langle w_i w_j \rangle = \langle w_i (x) w_j (0) \rangle \). This result is consistent with the appearance of \( \langle w_i w_j \rangle \) in the relations derived by von Karman and Howarth. The integral representation formula for \( \langle u_i u_j \rangle \) is, in a sense, opposite to the differential relations derived by Batchelor\(^6\) giving \( \langle u_i u_j \rangle \) in terms of second-order derivatives of the two-point velocity correlation tensor, \( R_{ij}(x, r) = \langle u_i (x) u_j (x + r) \rangle \).

The coarse graining hypothesis represents one approach toward solving the general problem of simplifying the exact representation formula for \( \langle u_i u_j \rangle \) so that it contains \( \langle w_i w_j \rangle \) and not \( \langle w_i w_j \rangle \). In Bernard\(^1\) this simplification was carried out by assuming that \( \langle w_i w_j \rangle \) had a top-hat profile with small support. This resulted in formulas that were similar to those derivable using the coarse graining hypothesis but with the addition of a term representing local effects. While this hypothesis as well as the coarse graining hypothesis have been used with some success in previous calculations,\(^4,5,6,7\) clearly, room exists for the development of more accurate methods of simplifying the representation formulas. One such approach is to develop better approximations to \( \langle w_i w_j \rangle \), which depend only on \( \langle w_i w_j \rangle \), and, possibly, additional parameters that may be computed from the statistical properties of the flow field.

In homogeneous isotropic turbulence the form of \( \langle w_i w_j \rangle \) is known, being in fact, similar to that of \( R_{ij}(x, r) \), with the exception that the longitudinal and transverse vorticity correlation coefficients replace the same quantities for the velocity field. In view of this, von Karman and Howarth observed that the isotropy conditions

\[
\langle u_{ij} u_{ij} \rangle = 2 \langle u_{ij} u_{ij} \rangle = -4 \langle u_{ij} u_{ij} \rangle = -4 \langle u_{ij} u_{ij} \rangle \quad (1)
\]

\( (i \neq j, \text{ no sum on } i \text{ or } j) \) can be checked by evaluating the integral representation formulas corresponding to the terms in (1) for given values of the vorticity correlation coefficients. Such an investigation has been carried out here as a first step in developing improved models of \( \langle w_i w_j \rangle \). In particular, in this study we will attempt to determine forms for the vorticity correlation coefficients which satisfy (1) as well as assess the degree of sensitivity of the representation formulas to the choice of vorticity correlation coefficients. In addition, the practicality of numerically integrating the representation formulas will be examined.

The computed results show that the values of the representation formulas are affected by moderate changes in the vorticity correlation coefficients. In particular it was discovered that of the forms of the vorticity correlation coefficients...
that were considered, only that corresponding to a Gaussian longitudinal velocity correlation coefficient satisfied all conditions. Some of the implications of this result in the study of homogeneous isotropic turbulence will be described below. In addition, the computed results suggest an approach that may be taken in developing useful forms of $W_{ij}(x, r)$ in shear flows. Finally, it will be seen that the costs of numerically evaluating the representation formulas are not so large as to be prohibitive.

In order to carry out this investigation it will first be necessary to derive integral representation formulas for $\langle u_{m} u_{n}\rangle$. A rough indication of how this can be accomplished was given previously. The relations derived there were incomplete, however, in that a full accounting was not made of the singularity in the integral representation of the velocity derivatives. In fact, a similar deficiency exists in the formulas derived by von Karman and Howarth. In the next section a complete set of integral representation formulas for $\langle u_{i} u_{i}\rangle$, $\langle u_{i} u_{j}\rangle$, and $\langle u_{m} u_{n}\rangle$ will be derived. In Sec. III a numerical method will be given for evaluating the representation formulas. Following this, in Sec. IV, the case of homogeneous isotropic turbulence will be studied numerically, and in the final section conclusions will be drawn.

II. INTEGRAL REPRESENTATION OF VELOCITY MOMENTS

Consider an incompressible fluid in turbulent motion occupying a domain $D$. Under wide conditions the fluctuating velocity field can be represented in terms of a fluctuating vector potential $\psi$ such that

$$u_{i} = \epsilon_{ijk} \psi_{j,i}, \quad (2a)$$

$$\psi_{i,i} = 0, \quad (2b)$$

where $\epsilon_{ijk}$ is the alternating tensor. In rectangular Cartesian coordinates $\psi_{i}$ is determined from

$$\nabla^{2} \psi_{i} = -u_{i}. \quad (3)$$

The solution to (3) may be written as

$$\psi_{i}(x) = -\int_{D} G_{i}(x, y) u_{i}(y) dy \quad \text{no sum on } i, \quad (4)$$

where $G_{i}(x, y)$ is a Green's function. Let $g(y - x)$ denote the singular part of $G_{i}(x, y)$ and $H_{i}(x, y)$ the nonsingular part. Then,

$$G_{i}(x, y) = g(y - x) + H_{i}(x, y), \quad (5)$$

where

$$g(y - x) = -1/(4\pi|x - y|), \quad (6)$$

and $H_{i}(x, y)$ is determined by $D$ and the boundary conditions on $\psi$. For a fixed boundary the tangential components of $\psi$ must be constant while the normal derivative of the normal component is zero. Since $G_{ij} = \partial G_{i}/\partial x_{j}$ is locally integrable at $x = y$, where it is singular, it follows that

$$\psi_{i,j}(x) = -\int_{D} G_{i,j}(x, y) u_{i}(y) dy \quad \text{no sum on } i. \quad (7)$$

On the other hand $G_{ij,i}$ is not locally integrable at $x = y$. Thus to compute $\psi_{i,j}$ the following approach may be taken instead. Substitute for $G_{i}$ in (4) using (5). This gives

$$\psi_{i}(x) = -\int_{D} g(y - x) u_{i}(y) dy$$

$$-\int_{D} H_{i}(x, y) u_{i}(y) dy \quad \text{no sum on } i. \quad (8)$$

The first term on the right-hand side may be written as

$$-\int_{R} g(y - x) u_{i}(y) dy + \int_{R \cdot D} g(y - x) u_{i}(y) dy, \quad (9)$$

where $u_{i}(y)$ is extended arbitrarily to $R$ so that the integrals in (9) exist. Changing variables $z = y - x$ in the first term in (9) gives

$$-\int_{R} g(z) u_{i}(z + x) dz. \quad (10)$$

Substitute (9) into (8) after using (10), and then compute $\partial^{2} / \partial x_{j} \partial x_{k}$. This gives

$$\psi_{i,j}(x) = -\int_{R} g(z) u_{i}(z + x) dz + \int_{R \cdot D} g_{ik}(y - x) u_{i}(y) dy$$

$$-\int_{D} H_{i,j}(x, y) u_{i}(y) dy. \quad (11)$$

The first term in the right-hand side of (11) may be written as

$$\lim_{\epsilon \to 0} -\int_{|z| > \epsilon} g(z) u_{i}(z + x) dz. \quad (12)$$

Define

$$J_{ij,m}(z, x) = g(z) u_{i}(z + x) M_{k} - u_{i}(z + x) J_{ik,m}(z, x). \quad (13)$$

Calculate $\partial / \partial x_{m}$ of (13) and sum. This gives

$$g(z) u_{i,j,k}(z + x) = w_{i}(z + x) g_{j,k}(z) + J_{ik,m}(z, x). \quad (14)$$

Integrate (14) over $|z| > \epsilon$ and apply the divergence theorem to obtain

$$\int_{|z| > \epsilon} g(z) u_{i,j,k}(z + x) dz$$

$$= \int_{|z| > \epsilon} w_{i}(z + x) g_{j,k}(z)$$

$$+ \int_{|z| = \epsilon} n_{m}(z) J_{ik,m}(z, x) dS(z), \quad (15)$$

where $n_{m}(z)$ is a unit normal vector (pointing inwards) to the sphere $|z| = \epsilon$, and $dS(z)$ is an element of surface area. A computation reveals that

$$\lim_{\epsilon \to 0} \int_{|z| = \epsilon} n_{m}(z) J_{ik,m}(z, x) dS(z) = w_{i}(x) \delta_{j,k} / 3. \quad (16)$$

Substituting (15) into (12), using (16) and placing the resulting expression in (11) gives

$$\psi_{i,j}(x) = -\lim_{\epsilon \to 0} \int_{D_{e}} G_{i,j}(x, y) u_{i}(y) dy$$

$$-w_{i}(x) \delta_{j,k} / 3 \quad \text{no sum on } i, \quad (17)$$

where $D_{e} = D - B_{e}$ and $B_{e}$ denotes the ball of radius $\epsilon$ cen-
tered at $x$. A contraction on $j$ and $k$ in (17) gives (3) since $G_{ij} = 0$ for $x \neq y$.

Substituting (7) into (2a) gives

$$u_j(x) = -\epsilon_{ik} \int_D G_{k,x}(x, y) u_i(y) dy. \tag{18}$$

Differentiating (2b) with respect to $x_i$ and using (17) gives

$$w_i(x) = -3 \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{i,x}(x, y) u_j(y) dy. \tag{19}$$

It then follows using (2a), (17), and (19) that

$$u_j(x) = -\epsilon_{im} \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{n,m}(x, y) u_n(y) dy + \epsilon_{ij} \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{p,n}(x, y) v_p(y) dy. \tag{20}$$

Define

$$A_{mp}^n = \left\langle \int_D G_{n,m}(x, y) u_n(y) dy \int_D G_{p,q}(x, z) v_q(z) dz \right\rangle, \tag{21}$$
$$B_{mp}^n = \left\langle \int_D G_{n,m}(x, y) u_n(y) dy \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{p,q}(x, z) u_q(z) dz \right\rangle, \tag{22}$$

and

$$C_{n,m}^{p,q} = \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{p,m}(x, y) v_p(y) dy \lim_{\epsilon \to 0} \int_{D^\epsilon} G_{n,q}(x, z) u_q(z) dz. \tag{23}$$

Squaring (18) and averaging gives

$$\langle u_i, u_j \rangle = \delta_{ij}(A_{mn}^{mm} - A_{mn}^{nn}) + A_{nn}^{nn} + A_{nn}^{mm} - A_{ij}^{mm} - A_{ij}^{nn}. \tag{24}$$

Multiplying (18) and (20) together and averaging gives

$$\langle u_i, u_{jk} \rangle = \delta_{ij}(B_{mkm}^{nn} - B_{mkm}^{mm} - B_{mkam}^{mn} + B_{mkam}^{nn}) + \delta_{ik}(B_{kmn}^{nm} - B_{kmn}^{mn} + B_{kmn}^{in} + B_{kmn}^{m} - B_{kmn}^{mn} - B_{kmn}^{km}. \tag{25}$$

Squaring (20) and averaging gives

$$\langle u_{im}, u_{jn} \rangle = \delta_{ij} \delta_{mn} - \delta_{ij} \delta_{mn} C_{n,m}^{p,q} + \delta_{ij} \left( C_{n,m}^{p,q} + C_{n,m}^{m,p} - C_{m,n}^{p,q} - 2C_{m,n}^{p,q} - C_{m,n}^{n,m} \right) + \delta_{im} \left( C_{n,m}^{p,q} + C_{n,m}^{m,p} - C_{m,n}^{p,q} + C_{m,n}^{p,q} \right) - \delta_{im} C_{n,m}^{p,q} + C_{m,n}^{p,q} + C_{m,n}^{p,q} - C_{m,n}^{p,q} + C_{m,n}^{p,q}. \tag{26}$$

For particular values of $i, j, k, m, n$, and $n$ the sums on the right-hand sides of (24)–(26) simplify considerably. Each of the terms $A_{mp}^n$, $B_{mp}^n$, and $C_{mnp}^n$ can be written in the general form

$$I = \lim_{\epsilon \to 0} I_{\epsilon}, \tag{27}$$

where

$$I_{\epsilon} = \left\langle \int_{D^\epsilon} \int_{D^\epsilon} dy A(x, y) w_i(y) \int_{D^\epsilon} dz B(x, z) w_j(z) \right\rangle. \tag{28}$$

and $A$ and $B$ represent any particular components of $G_{ij}$ or $G_{ijk}$. Commuting the averaging operator in (28) with the integrals gives

$$I_{\epsilon} = \int_{D^\epsilon} dy A(x, y) \int_{V_{ij\epsilon}} dz B(x, z) W_{ij}(y, z - y). \tag{29}$$

where $V_{ij}(y) = V_{ij}(y) D_\epsilon$ and $V_{ij}(y) = \{ z; W_{ij}(y, z - y) = 0 \}$, see Fig. 1. The appearance of $W_{ij}(y, z)$ in (29), and hence in (24)–(26), coincides with the similar result of von Karman and Howarth.\(^3\)

Squaring (19) and averaging and applying (23) gives

$$\langle w_i w_j \rangle = W_{ij}(x, 0) = 9 C_{m,n}^{m,n}. \tag{30}$$

Using (20) it may be seen that (19) is equivalent to $u_i = \epsilon_{ij} u_{kj}$. Hence (30), in fact, expresses the identity

$$\langle w_i w_j \rangle = \epsilon_{im} \epsilon_{jn} \langle u_{in} u_{jp} \rangle. \tag{31}$$

Equation (30) is an integral equation that $W_{ij}(x, y)$ must satisfy in any turbulent flow field. In general it is not expected.
to have a unique solution since it is a purely kinematical condition.

III. NUMERICAL EVALUATION OF THE INTEGRALS

For given values of \( W_y \) the numerical integration of the terms in (24)–(26) can be carried out using (27) and (29). Here \( I_i \) may be evaluated using any one of several numerical integration techniques.\(^{12-14} \) For the present study a three-dimensional rectangle rule was used since it is relatively easy to program for the very large number of double volume integrals that are contained in (24)–(26).

For the calculations presented in the next section, the mesh used in evaluating the outer integral in (29) consisted of boxes with centers at \( y_{jmn} = x + d_m d_n \), where \( d_m = \{d^i, d^i_m, d^i_m \}, d^i_m = j/3^i + 1/3^i \), \( i = 1 \rightarrow N_x \), \( j = 1 \rightarrow N_y \), \( m = 1 \rightarrow N_z \), and \( 3N_x, 3N_y, \) and \( 3N_z \) are positive integers, and \( N_x, N_y, \ldots \) are integers satisfying \( N_x - 1 < 3N_x \) for \( i = 2 \rightarrow N_x \). Figure 2 shows the form of this grid in two dimensions for \( x = 0, N_x = 3 \), and \( N_y = N_z = N_x = 1 \). The inner integral in (29) is summed on the points \( y_{jmn} = x + d_m d_n, k = 1 \rightarrow N_x, p = P^\perp_k \rightarrow P^\perp_k, q = Q^\perp_k \rightarrow Q^\perp_k, \) and \( r = R^\perp_k \rightarrow R^\perp_k \), where \( P^\perp_k, Q^\perp_k, \) \( Q^\perp_k, R^\perp_k \), and \( R^\perp_k \) are integers depending on \( i, j, m, \) and \( n \) in the outer integral. The points \( y_{jmn} \) cover just the support of \( W_y \). Their density varies in accordance with the local density of grid points used in the outer integral. The rectangle rule approximation to (29) on this mesh is

\[
I_i \approx \left( \frac{1}{3^{N_x + 1 - i}} \right) \sum_j \sum_m \sum_n A(x, y_{jmn}) \left[ \sum_k \left( \frac{1}{3^{N_x + 1 - k}} \right) \sum_p \sum_q B(x, y_{jmn}^k) \right] W_y(y_{jmn}^k, z_{jmn}^k - y_{jmn}^k) \tag{32}
\]

where all those values of \( j, m, \) and \( n \) or \( p, q, \) and \( r \) for which \( y_{jmn} \) or \( z_{jmn} \) has appeared in a lower value of \( i \) or \( k \) are excluded. The point \( i = 1, j = m = n = 0 \) is excluded from the outer sum in (32) as is the grid box in the inner sum which contains the singular point \( y_{jmn} \) if \( x \) should occur. By increasing \( N_i \) the mesh becomes more refined near \( x \). The box containing \( x \) is of linear dimension \( 1/3^N \). Since this box is excluded in the inner and outer sums, the integration is essentially carried out over the regions \( V_e \) as required. When the mesh is sufficiently refined, the sums in (32) must be approximately independent of \( N_x, N_y, P^\perp_k, Q^\perp_k, Q^\perp_k, R^\perp_k, \) and \( R^\perp_k \). Generally this occurs when the support of \( W_y \) is significantly greater than the size of the excluded box.

The region covered by the outer sum is a cube with sides of length \( 2N_x + 1 \). In principle \( N_x \) is chosen large enough so that the entire flow domain is contained in the outer mesh. In practice this is not necessary since the contributions to \( I_i \) from grid boxes relatively far from \( x \) are quite small and need not be included in the summation.

IV. APPLICATION TO HOMOGENEOUS ISOTROPIC TURBULENCE

Consider fluid in homogeneous isotropic turbulent motion occupying a region of space \( D \). At any fixed time \( \langle u^2 \rangle \) and \( \langle w^2 \rangle \), the mean square velocity and vorticity components, respectively, are uniform throughout the flow field. In general \( \langle u^2 \rangle \) and \( \langle w^2 \rangle \) vary in time due to dissipation effects or the presence of a uniform source of turbulence. The components \( \langle u^2 \rangle \) and \( \langle w^2 \rangle \) define a characteristic length scale of the flow field, \( \lambda = \langle \tau^2 \rangle / \langle u^2 \rangle \), where \( \tau \) is the Taylor microscale. Using (1) it follows that \( \langle u^2 \rangle \) and \( \langle w^2 \rangle \) have the identity (when \( i = j = 3 \)),

\[
\langle w^2 \rangle = \langle u^2 \rangle + 2 \langle u_3 u_{3,1} \rangle \tag{34}
\]

When \( i = j \), (30) gives the condition

\[
1 = 9(C_{1111}^1 + 2C_{1113}^1 + 2C_{1213}^3 + 4C_{1223}^3, \tag{33}
\]

where \( C_{max} \) is assumed here and in the sequel to be nondimensionalized by \( \langle u^2 \rangle \). In obtaining (33) the symmetry properties of homogeneous isotropic turbulence were used to group like terms. According to (31), (33) expresses the identity (when \( i = j = 3 \)),

\[
\langle w^2 \rangle = \langle u^2 \rangle + 2 \langle u_3 u_{3,1} \rangle \tag{34}
\]

For \( i = j \), (30) gives

\[
0 = 2C_{1112}^1 + C_{1123}^3 + C_{2333}^3 + C_{3123}^3 + 2C_{1223}^3 + 2C_{2313}^3, \tag{35}
\]

each of whose terms may be shown to be identically zero because of symmetry. Thus (35) need not be considered further.

Equations (1) and (34) imply that

\[
\langle w^2 \rangle = 2 \langle u_{3,2} \rangle \tag{36a}
\]

\[
\langle w^2 \rangle = 2 \langle u_{3,1} \rangle \tag{36b}
\]

\[
\langle w^2 \rangle = -10 \langle u_{3,1} u_{3,1} \rangle \tag{36c}
\]

\[
\langle w^2 \rangle = -10 \langle u_{4,2} \rangle \tag{36d}
\]

Using (26), (36a)–(36d) give the conditions (37)–(40), respectively:

FIG. 2. Computational mesh with \( N = 3, N_i = 1, N_z = 1, \) and \( N_i = 1 \).

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\[ 1.0 = \frac{3}{7} C_{111}^{222} - 2 C_{113}^{113} + 5 C_{1311}^{111} + C_{1111}^{111} - 2 C_{1122}^{222} + 2 C_{1222}^{222} + 4 C_{1211}^{111} \]  
\[ 1.0 = 5(2C_{1122}^{112} - 2 C_{1132}^{113}) \]  
\[ 1.0 = 10( - C_{1122}^{112} - 4 C_{1132}^{113} - 2 C_{1211}^{111} + 2 C_{1222}^{222} - 5 C_{1311}^{111} - 8 C_{2222}^{222}) \]  
\[ 1.0 = -10( - C_{1122}^{112} + C_{1132}^{113}). \]  

Since (36c) may be derived from (34) and (36a) it follows that (33), (37), and (39) are linearly dependent, as a direction calculation also shows. Consequently (39) may be eliminated from further consideration. In addition, (40) is redundant since it is identical to (38). This is a consequence of (1) and the continuity equation. The two additional conditions,

\[ 1 = 10[A_{11}^{11} - A_{12}^{12}], \]

\[ 0 = -A_{11}^{11} + 2A_{12}^{12} - A_{12}^{12}, \]

where \( A_{\text{iso}} \) is nondimensionalized by \( \langle u^2 \rangle \lambda^2 \), are a consequence of the fact that \( \langle u_t u_r \rangle = \langle u^2 \rangle \delta_{r\theta} \) in homogeneous isotropic turbulence. As in (35), each of the terms in (42) are zero because of symmetry and thus (42) need not be considered further.

The preceding discussion has provided four independent conditions (33), (37), (38), and (41) which must be satisfied by \( W_\theta(x, r) \) in homogeneous isotropic turbulence. The isotropic form of \( W_\theta(x, r) \) is

\[ W_\theta(x, r) = \langle u^2 \rangle \left\{ [C(r) - D(r)] r_i r_i / r^2 + D(r) \delta_{00} \right\}, \]  

where \( r = |r| \) and \( C(r) \) and \( D(r) \) are the longitudinal and transverse vorticity correlation coefficients. A relation similar to (43) also holds for \( R_\theta(x, r) \) with \( F(r) \), the longitudinal, and \( G(r) \), the transverse, velocity correlation coefficients replacing \( C(r) \) and \( D(r) \), respectively. Since \( u_t \) and \( w_i \) are solenoidal, \( G(r) \) may be obtained from \( F(r) \), and \( D(r) \) from \( C(r) \), using well-known relations. \( E(k) \), the energy spectrum function, may be calculated from \( F(r) \) using the appropriate transform. Similarly \( C(r) \) may be computed from \( \Omega(k) \), the squared vorticity spectrum. Since \( \Omega(k) = k^3 E(k) \), it follows that \( C(r) \) may be calculated from given values of \( F(r) \) by computing \( E(k) \) as an intermediate step. Alternatively, \( C(r) \) may be calculated directly from \( F(r) \) using the differential relation given by Batchelor.\(^6\)

The velocity correlation coefficient \( f(r/\lambda) \) has been determined experimentally in the approximately homogeneous isotropic turbulent flow downstream of a grid placed in a uniform flow field.\(^15\)-\(^18\) In addition, forms for \( f(r/\lambda) \) in isotropic turbulence have been derived from theoretical considerations.\(^3,5,15,20\) In contrast, forms of \( c(r/\lambda) = C(r) \) are generally unavailable. Consequently, given values of \( f(r/\lambda) \) were used to generate forms of \( c(r/\lambda) \) to be used in the present study. This was done by numerically evaluating the transform relations between \( F(r) \) and \( E(k) \), and \( \Omega(k) \) and \( C(r) \) using the Filon–Spline\(^2\) and the trapezoidal rules, respectively. The accuracy of the computed curves for \( c(r/\lambda) \) were verified by varying the mesh size and by back substituting the calculated values of \( E(k) \) and \( C(r/\lambda) \) to recover \( f(r/\lambda) \) and \( \Omega(k) \).

Here \( f(r/\lambda) \) must satisfy the conditions \( f(0) = 1 \), \( f'(0) = 0 \), and \( f''(0) = -1 \), where \( f' = df/da \). A convenient method of obtaining a family of functions satisfying these conditions is to solve the initial value problem

\[ f''(s) + (4/\alpha + s/\alpha^2) f'(s) + 5 f(s) = 0, \]  

\[ f(0) = 1, \quad f'(0) = 0, \]  

for different values of the parameter \( \alpha \). Equation (44a), for \( s = r/\lambda \), is the similarity relation derived by von Karman and Howarth.\(^3\) All solutions to (44) satisfy \( f''(0) = -1 \). When \( \alpha = 0.25 \) the solution to (44) is

![FIG. 3. Longitudinal velocity coefficients \( f(r/\lambda) \) considered in this study.](image)
\[
f(r/\lambda) = \exp[-(r/\lambda)^2/2],
\]
which is the value of \( f \) corresponding to the final period of decay.\(^{17} \) Since this flow field is perhaps the most likely to be fully isotropic among those which have been considered experimentally, the \( f(r/\lambda) \) curves used in the present study were chosen to be grouped around the Gaussian curve. The five curves shown in Fig. 3 were considered in this study. Those labeled \( \alpha = 0.05, 0.1, 0.25, \) and 1.0 correspond to solutions of (44). The remaining curve was measured experimentally by Stewart and Townsend\(^{18} \) at a location \( x/M = 90 \) down-stream of a grid with mesh Reynolds number \( R_M = 5300. \) Figure 4 shows the \( c(r/\lambda) \) curves corresponding to the curves in Fig. 3. For \( \alpha = 0.25, \) a direct calculation gives
\[
c(r/\lambda) = \left[1 - (r/\lambda)^2/5\right] \exp[-(r/\lambda)^2/2],
\]
and
\[
d(r/\lambda) = \left[1 - \frac{\gamma}{\eta_0}(r/\lambda)^2 + \frac{\gamma}{\eta_0}(r/\lambda)^4\right] \times \exp[-(r/\lambda)^2/2].
\]

Table I contains computed values of the right-hand sides of (33), (37), (38), and (41) corresponding to the five curves in Fig. 4. All of these calculations were done using the identical mesh. The values in Table I show that \( W_0 \) corresponding to \( \alpha = 0.25 \) comes closest to satisfying all of the conditions while \( W_0 \) arising from the data of Stewart and Townsend does so the least. The computed results generally get closer to 1.0 as \( \alpha = 0.25 \) is approached.

Tables II and III show the results of a series of computations for \( \alpha = 0.25 \) and 0.05, respectively, on progressively finer and more extensive meshes. The first two lines in Table II show that lowering \( \epsilon \) from \( \frac{1}{10} \) to \( \frac{1}{50} \) had only a very small effect on the computed values. Thus \( \frac{1}{50} \) may be taken to be a sufficiently small value of \( \epsilon. \) The first and third lines in Table II show that increasing the size of the computational domain from \( N_2 = 2 \) to 3 for \( \alpha = 0.25 \) gave only small additional contributions to the sums. For \( \alpha = 0.05 \) the first three lines of Table III show that at \( N_2 = 3, \) (33), (37), and (38) have stabilized while (41) has nearly done so. An additional check was made of the change of (41) with increasing \( N_2 \) which showed that no significant far-field contributions to (41) are neglected by not using a larger value of \( N_2. \) The approximate times of the calculations are also given in Tables II and III. Different timings for identical values of \( N, N_2, \) and \( N_1 \) reflect the use of meshes of different size and refinement in the inner sums in (32).

It may be seen from Table II that the computed values for \( \alpha = 0.25 \) are reasonably close to satisfying all constraints. For the last computation, which took 20 minutes on the University of Maryland UNIVAC 1180 computer, the calculated values differ from 1.0 by 0.68%, 0.06%, 2.52%, and 2.22%, respectively. Hence, it may be concluded

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**Table I.** Computed values of Eqs. (33), (37), (38), and (41) for the curves of Fig. 4, using a mesh with \( N = 2, N_1 = 1, \) and \( N_2 = 2. \)

<table>
<thead>
<tr>
<th>Curve</th>
<th>Eq. (33)</th>
<th>Eq. (37)</th>
<th>Eq. (38)</th>
<th>Eq. (41)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stewart and Townsend</td>
<td>0.9475</td>
<td>0.9475</td>
<td>0.8621</td>
<td>0.7079</td>
</tr>
<tr>
<td>( \alpha = 0.05 )</td>
<td>0.9612</td>
<td>0.9636</td>
<td>0.9024</td>
<td>0.7389</td>
</tr>
<tr>
<td>( \alpha = 0.1 )</td>
<td>0.9605</td>
<td>0.9682</td>
<td>0.9126</td>
<td>0.90</td>
</tr>
<tr>
<td>( \alpha = 0.25 )</td>
<td>0.9698</td>
<td>0.9844</td>
<td>0.9321</td>
<td>1.0220</td>
</tr>
<tr>
<td>( \alpha = 1.0 )</td>
<td>0.9943</td>
<td>1.0116</td>
<td>0.9058</td>
<td>1.0699</td>
</tr>
</tbody>
</table>
that a Gaussian form for \( f(r/\lambda) \), or one quite close to the Gaussian, is fully consistent with all of the constraints. In contrast, the last line of Table III shows that for an equivalent calculation with \( \alpha = 0.05 \) the computed values of (33), (37), (38), and (41) differed from 1.0 by 0.68%, 1.23%, 7.29%, and 18.63%, respectively. Thus it does not appear that the \( \alpha = 0.05 \) curve is consistent with the isotropy conditions. This, plus the trends evident in Table I suggest that a Gaussian may be the only permissible form for \( \dot{f} \). However, this cannot be said with certainty at the present time in view of the small number of \( f(r/\lambda) \) curves which have been tested thus far.

Many experimentally determined forms of \( f(r/\lambda) \) display a behavior similar to the curve found by Stewart and Townsend in Fig. 3, i.e., a relatively slow dropoff of \( f \) with increasing \( r \). The preceding calculations suggest that these curves will not satisfy the isotropy conditions. However, it is known that the laboratory flows that are generally used as representative of isotropic turbulence fail to display complete isotropy. In particular, the low wavenumber end of the spectrum is anisotropic which can result in correlation between velocities at widely separated points. Thus the present computations may be interpreted as providing additional evidence suggesting that the turbulent fields which have been used in obtaining correlation functions, such as that of Stewart and Townsend in Fig. 3, are not isotropic.

Since the computed values of (33) given in Tables I–III are relatively close to 1.0 for all \( W_0 \), it appears that (33) does not have a unique solution. As mentioned previously, such a result is not unexpected.

If it were true that (45) is the only curve that is fully consistent with the isotropy conditions (33), (37), (38), and (41), then clearly \( f \) would be self-preserving throughout the decay of isotropic turbulence. Batchelor and Townsend\(^\text{17}\) showed that a self-preserving Gaussian form for \( f \) occurs in the final period of decay, when inertial effects are negligible. However, nothing in the present kinematical analysis has placed a restriction on when in the decay process (45) might hold. Thus, it might be profitable to investigate the existence of self-preserving solutions to the isotropic decay problem in which (45) holds at all times. Apparently, this case has not been considered in previous investigations of self-preserving isotropic turbulent flows.\(^\text{15,18–20}\)

The self-preserving form of the von Karman–Howarth equation governing the decay of isotropic turbulence is\(^\text{1,5}\)

\[
\dot{f}(s) + \frac{4}{s} f'(s) + 5 f(s) + \frac{\lambda \dot{\lambda}}{2s^2} f''(s) + \frac{1}{2s^2} v \lambda \dot{\lambda} \left( \frac{k}{s} + \frac{4k}{s^2} \right) = 0,
\]

where \( s = r/\lambda, \nu = (\nu^2)^{1/2}, \nu \) is the kinematic viscosity, \( k = k(r/\lambda) = K(r) \) is the triple velocity correlation coefficient, \( \lambda = d\lambda/dt \), and \( t \) represents time. Substituting \( f'(s) = \exp(-s^2/2) \) in (48) and separating terms, gives

\[
\frac{k' + 4k}{s} \exp(-s^2/2) = \frac{\lambda \dot{\lambda}}{2s^2} \frac{1}{2s^2} v \lambda \dot{\lambda} \cdot
\]

in which the left side depends only on \( s \) and the right side only on \( t \). Thus both sides of (49) must be equal to a separation constant, say \( -\delta \). This yields the following two equations:

\[
k' + 4k = -\delta s^2 \exp(-s^2/2),
\]

and

\[
\lambda \dot{\lambda} = 2
\]

Equation (50) may be integrated directly yielding

\[
k\left( \frac{r}{\lambda} \right) = -\delta \frac{r^4}{4} \int_0^r x^2 \exp\left( -\frac{x^2}{2} \right) dx.
\]

As \( r \to \infty \), (52) implies that

\[
k \sim r^{-4},
\]

in agreement with the deductions of Proudman and Reid\(^\text{22}\) and Batchelor and Proudman.\(^\text{23}\) As \( r \to 0 \), (52) shows that

\[
k \sim r^4,
\]

which agrees with the discussion given by Hinze.\(^\text{15}\)

Equation (51) contains two unknowns, \( \nu \) and \( \lambda \). However another relationship between these quantities is furnished by the turbulent kinetic energy equation\(^\text{15}\)

\[
\nu^2 = 10 \nu^2 / \lambda.
\]

If \( \nu \) and \( \dot{\nu} \) in (53) are replaced using (51) then the following equation for \( \lambda \) alone results:

\[
\lambda \frac{d^2 \lambda}{dt^2} + 7 \nu \lambda \dot{\lambda} - 10 \nu = 0.
\]

Let \( \lambda_0 \) and \( \nu_0 \) be the initial values of \( \lambda \) and \( \nu \) in the decay problem. Here \( R = \lambda_0 \nu_0 / \nu \) is the initial value of the turbulence Reynolds number. If (54) is nondimensionalized using \( \lambda_0 \) and \( \nu_0 \) then the following initial value problem for \( \lambda(t) \) results:
\[ \lambda^3 \lambda + 7 \lambda \lambda / R - 10 / R^2 = 0, \quad (55a) \]
\[ \lambda(0) = 1, \quad (55b) \]
\[ \dot{\lambda}(0) = -\delta + 2 / R, \quad (55c) \]
where (55c) follows from (51). The dimensionless form of (53) may be integrated to give
\[ \psi(t) = \exp \left( \frac{-5}{R} \int_0^t \frac{1}{\lambda_0(s)} ds \right). \quad (56) \]
Thus (55) and (56) furnish a complete solution to the isotropic decay problem for each value of \( \delta \) and \( R \). When \( \delta = 0 \), (52) implies that \( k(r/\lambda) = 0 \), and (51) gives \( \lambda^2 = 4 \psi t + C_1 \), where \( C_1 \) is a constant. Thus this case corresponds to the final period of decay. The nature of the solutions to (55) and (56) for \( \delta \neq 0 \) will be the subject of a subsequent paper dealing with the dynamics of the decay process in isotropic turbulence.

Parenthetically, it may be remarked that the analysis leading to (49) serves as a counterexample to the deductions made by Hinze (see Ref. 15, p. 265) concerning the conditions that all solutions to (48) must satisfy.

V. CONCLUSIONS

Approximate forms of (24)–(26) may be used in vorticity-based closures to calculate velocity moments. Table I shows that relatively moderate changes in the form of \( W_y \) can have a significant effect on the computed values of the integral terms in (24)–(26). Thus, for example, if a crude assumption such as the top-hat profile is used to model \( W_y(x, r) \), then it is likely that the computed values of the velocity moments will be significantly in error. At a minimum, whatever choice of \( W_y \) is used must satisfy (30) at every fixed time in a turbulent flow calculation. This may be accomplished by using forms of \( W_y \) which contain parameters that are determinable from (30).

Since the form of \( W_y \) supplied by (43), (46), and (47) apparently satisfies all constraints in isotropic turbulence, it may be taken as a starting point for the development of forms of \( W_y \) appropriate to shear flows. In particular, more general forms of \( W_y \) applicable to anisotropic and inhomogeneous turbulence may be derived as perturbations to this form of \( W_y \). These may depend on the aforementioned parameters which would be determined from (30). In flows for which measurements of correlations such as \( \langle u_i u_j \rangle \) are available, \( \langle w_i w_j \rangle \) may be treated as an unknown in the assumed model of \( W_y \) and solved for, using (24). Once \( \langle w_i w_j \rangle \) (and \( W_y \)) is determined for such a flow it can then be used to calculate the global values of correlations such as \( \langle u_i, u_j k \rangle \) and \( \langle u_i n, u_j n \rangle \), which are at present difficult to measure. Currently, an investigation of the turbulent flow in a channel is being carried out using this approach. The forms of \( W_y \) developed in this way may also be applied to general flows for which measurements of \( \langle u_i u_j \rangle \) are unavailable.

Finally, the present calculations show that the large amount of computations implicit in the use of (24)–(26) are not prohibitive. Tables I–III indicate that relatively coarse grids can be used for the calculations without too great a loss of accuracy. In practical computations (24)–(26) can be approximated in the form of moving averages with coefficients that are determined once and for all time and stored. In the present computations the outer integral in (24)–(26) had to be summed over a region of approximate radius \( 3 \delta \) in order to achieve convergence. Similarly, the inner sums required a region of radius \( \approx 3 \lambda \). A typical value of \( \lambda \) in the central region of a pipe or channel flow is \( \approx 0.05 D / 2 \), where \( D \) is the pipe diameter or channel width. Hence, the support of \( W_y \) may be assumed to have a radius \( \approx 0.15 D / 2 \) and the total region to be included in computing the velocity moments has radius \( \approx 0.25 D / 2 \). Near the wall \( \lambda \) has even smaller values. Thus it may be expected in typical applications of (24)–(26) that the summations need only be carried out over a local region considerably smaller than the whole flow domain, thereby reducing the numerical expense considerably.

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