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The energy decay in self-preserving isotropic turbulence revisited

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The assumption of self-preservation permits an analytical determination of the energy decay in isotropic turbulence. Batchelor (1948), who was the first to carry out a detailed study of this problem, based his analysis on the assumption that the Loitsianskii integral is a dynamic invariant – a widely accepted hypothesis that was later discovered to be invalid. Nonetheless, it appears that the self-preserving isotropic decay problem has never been reinvestigated in depth subsequent to this earlier work. In the present paper such an analysis is carried out, yielding a much more complete picture of self-preserving isotropic turbulence. It is proven rigorously that complete self-preserving isotropic turbulence admits two general types of asymptotic solutions: one where the turbulent kinetic energy $K \sim t^{-1}$ and one where $K \sim t^{-\alpha}$ with an exponent $\alpha > 1$ that is determined explicitly by the initial conditions. By a fixed-point analysis and numerical integration of the exact one-point equations, it is demonstrated that the $K \sim t^{-1}$ power law decay is the asymptotically consistent high-Reynolds-number solution; the $K \sim t^{-\alpha}$ decay law is only achieved in the limit as $t \rightarrow \infty$ and the turbulence Reynolds number R_t vanishes. Arguments are provided which indicate that a t^{-1} power law decay is the asymptotic state toward which a complete self-preserving isotropic turbulence is driven at high Reynolds numbers in order to resolve an $O(R_t^{\frac{1}{2}})$ imbalance between vortex stretching and viscous diffusion. Unlike in previous studies, the asymptotic approach to a complete self-preserving state is investigated which uncovers some surprising results.

1. Introduction

Despite the fact that isotropic turbulence constitutes the simplest type of turbulent flow, it is still not possible to render the problem analytically tractable without the introduction of additional hypotheses. The idealization of self-preservation – wherein the two-point double and triple longitudinal velocity correlations are assumed to admit self-similar solutions with respect to a single lengthscale $L(t)$ – has served as a useful hypothesis since its introduction by von Kármán & Howarth (1938). In another classic paper that followed, Batchelor (1948) studied the energy decay in self-preserving isotropic turbulence in considerable detail. He concluded that the only complete self-preserving solution that was internally consistent existed at low turbulence Reynolds numbers where the turbulent kinetic energy $K \sim t^{-\frac{5}{2}}$ – a power law consistent with the final period of decay. Batchelor (1948) also found a self-preserving solution to the Kármán–

Howarth equation in the limit of infinite Reynolds numbers for which Loitsianskii's integral was an invariant. This solution – wherein $L(t)$ is the integral lengthscale Λ and $K \sim t^{-\frac{10}{7}}$ – was put forth by Batchelor as the only full self-preserving solution at high Reynolds numbers. Of course, additional partial self-preserving solutions were shown by Batchelor to exist in other Reynolds-number regimes.

Objections were later raised to the use of the Loitsianskii integral as a dynamic invariant: at high Reynolds numbers this integral can be shown to be a weak function of time (see Proudman & Reid 1954 and Batchelor & Proudman 1956). Saffman (1967) proposed an alternative dynamic invariant which yields a $K \sim t^{-\frac{6}{5}}$ power law decay in the infinite-Reynolds-number limit (see Hinze 1975). While the results of Batchelor and Saffman formally constitute complete self-preserving solutions to the inviscid Kármán–Howarth equation, it must be kept in mind that they only exhibit partial self-preservation with respect to the full viscous equation. Namely, there is self-preservation only for the range of energy-containing eddies with integral lengthscale Λ (here, $\lambda/\Lambda \ll 1$ where λ is the Taylor microscale). These two solutions have been widely accepted in the turbulence literature as the predicted decay laws for self-preserving isotropic turbulence at high Reynolds numbers.

Implicit in the analysis of Batchelor (1948) is the existence of a complete self-preserving solution consistent with high Reynolds numbers – namely a $K \sim t^{-1}$ power law decay. The collapsing lengthscale $L(t)$ for this full self-preserving solution is necessarily the Taylor microscale (i.e. for any complete self-preserving solution of the viscous Kármán–Howarth equation we must have $L \propto \lambda$). This solution – which was postulated a few years earlier by Dryden (1943) – was dismissed by Batchelor on the grounds that Loitsianskii's integral was not a dynamic invariant therein. While the result by Dryden has been mentioned subsequently in the literature (cf. Hinze 1975; Monin & Yaglom 1975 and Korneyev & Sedov 1976), it has largely been disregarded by the turbulence community. The reason for this appears to be two-fold: (*a*) a $K \sim t^{-1}$ power law decay has not been observed in the most accurate isotropic decay experiments, and (*b*) since $\lambda/\Lambda \rightarrow 0$ as $Re \rightarrow \infty$, questions can be raised about the suitability of the Taylor microscale as the collapsing lengthscale of the energy-containing eddies.

Recently, George (1987, 1989, 1992) revived this issue concerning the existence of complete self-preserving solutions in isotropic turbulence. In an interesting paper he claimed to find a complete self-preserving solution, *valid for all Reynolds numbers*, in which the kinetic energy decayed as $K \sim t^{-\alpha}$ with α determined by the initial conditions. George (1987) – who based his analysis on the dynamic equation for the energy spectrum rather than on the Kármán–Howarth equation – made no explicit mention of the complete self-preserving $K \sim t^{-1}$ solution. Strictly speaking, the solution presented by George was an alternative self-preserving solution to that of Kármán & Howarth (1938) and Batchelor (1948) since he relaxed the constraint that the triple longitudinal velocity correlation be self-similar in the classical sense.

The purpose of the present paper is to address the issue of complete self-preservation in an effort to clarify the following basic questions.

(i) What is the complete self-preserving solution for isotropic turbulence at high Reynolds numbers?

(ii) What detailed predictions does this solution yield for the energy decay, particularly during the initial approach to a state of complete self-preservation?

(iii) Is this solution compatible with the results of physical experiments and alternative theoretical approaches?

In so far as the first two points are concerned, it will be shown unequivocally that the only complete self-preserving solution that is consistent with a high-Reynolds-number isotropic turbulence has a $K \sim t^{-1}$ asymptotic power law decay. Unlike previous studies, this is demonstrated in a straightforward manner based on a fixed-point analysis of the one-point equations. This analysis leads to an interesting interpretation of the physical significance of a $K \sim t^{-1}$ power law decay and allows us to examine small departures from a state of complete self-preservation. The detailed predictions of this complete self-preserving solution – which, to the best of our knowledge, have never been examined in depth in the literature – will be compared with the results of physical experiments and alternative theoretical approaches in the sections to follow.

2. Theoretical background

We will consider isotropic turbulence governed by the incompressible Navier–Stokes equations

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}+\nu\nabla^2 u_i, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2}$$

where u_i is the velocity vector, p is the pressure and ν is the kinematic viscosity. The two-point double and triple longitudinal velocity correlations, denoted by $f(r,t)$ and $k(r,t)$, respectively, are defined in the standard way:

$$f(r,t) = \frac{\overline{u(\boldsymbol{x},t)\,u(\boldsymbol{x}+\boldsymbol{r},t)}}{\overline{u^2}}, \tag{3}$$

$$k(r,t) = \frac{\overline{u^2(\boldsymbol{x},t)\,u(\boldsymbol{x}+\boldsymbol{r},t)}}{(\overline{u^2})^{\frac{3}{2}}}, \tag{4}$$

where u is any component of the velocity, $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$ are any two spatial points separated by a distance $r = |\boldsymbol{r}|$ in the direction of u, and an overbar denotes a spatial average (cf. Hinze 1975). For isotropic turbulence, f and k satisfy the Kármán–Howarth equation

$$\frac{\partial(\overline{u^2}f)}{\partial t} = (\overline{u^2})^{\frac{3}{2}}\left(\frac{\partial k}{\partial r}+\frac{4}{r}k\right)+2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{5}$$

which is obtained directly from the Navier–Stokes equations. The turbulent kinetic energy $K \equiv \frac{1}{2}\overline{u_i u_i}$ is a solution of the differential equation

$$\dot{K} = -\epsilon, \tag{6}$$

where

$$\epsilon = \nu\overline{\omega_i\,\omega_i} \equiv \nu\omega^2 \tag{7}$$

is the turbulent dissipation rate, ω_i is the vorticity vector and ω^2 is the enstrophy. The turbulent dissipation rate is a solution of the differential equation

$$\dot{\epsilon} = \frac{7}{3\sqrt{15}}S_K R_t^{\frac{1}{2}}\frac{\epsilon^2}{K}-\frac{7}{15}G\frac{\epsilon^2}{K}, \tag{8}$$

where
$$S_K = -\frac{\overline{(\partial u/\partial x)^3}}{[\overline{(\partial u/\partial x)^2}]^{\frac{3}{2}}} = -\lambda^3\left[\frac{\partial^3 k}{\partial r^3}\right]_{r=0}, \tag{9}$$

$$G = \lambda^4\left[\frac{\partial^4 f}{\partial r^4}\right]_{r=0}, \tag{10}$$

$$R_t = \frac{K^2}{\nu\epsilon}, \quad \lambda = \left(\frac{10\nu K}{\epsilon}\right)^{\frac{1}{2}} \tag{11}$$

are the velocity derivative skewness, the coefficient of the enstrophy destruction, the turbulence Reynolds number, and the Taylor microscale, respectively. Equations (6) and (8) – which are rearranged versions of those that appear in Kármán & Howarth (1938) and Batchelor (1948) – are obtained, respectively, by setting $r = 0$ in (5) and in the equation obtained by differentiating (5) twice with respect to r. Since $S_K = S_K(t)$ and $G = G(t)$ are directly related to the correlations f and k (which cannot both be obtained from the Kármán–Howarth equation (5)) it is clear that the problem of isotropic decay is not closed. In order to achieve closure, additional assumptions must be made such as the one of complete self-preservation that will be discussed in this paper.

For an isotropic turbulence to be self-preserving in the sense of Kármán & Howarth (1938) and Batchelor (1948), we must have

$$f(r,t) = \tilde{f}(r/L), \tag{12}$$

$$k(r,t) = \tilde{k}(r/L), \tag{13}$$

where $L = L(t)$ is a uniquely specified similarity lengthscale. For it to exhibit complete self-preservation, all scales of the turbulence – namely, the full range of $0 \leqslant r \leqslant \infty$ – must decay according to (12) and (13); partial self-preservation is satisfied if (12) and (13) only apply to some restricted range of $0 \leqslant r \leqslant r_{max}$. We will focus our attention on complete self-preserving solutions in the analysis to follow. In view of the identity (Batchelor 1948)

$$\epsilon = -10\nu K\left[\frac{\partial^2 f}{\partial r^2}\right]_{r=0} \tag{14}$$

it follows from (11) that

$$\lambda^2\left[\frac{\partial^2 f}{\partial r^2}\right]_{r=0} = -1. \tag{15}$$

Hence, for any complete self-preserving isotropic turbulence, we must have

$$\frac{\lambda^2}{L^2}\tilde{f}''(0) = -1, \tag{16}$$

from which it can be concluded that

$$L \propto \lambda \tag{17}$$

since $\tilde{f}''(0)$ is a constant. It therefore follows that *the Taylor microscale is the only similarity lengthscale that can yield complete self-preserving solutions to the full viscous equations of motion for isotropic turbulence.*

Without any loss of generality for a complete self-preserving isotropic turbulence we may set $L = \lambda$ and then substitute (12)–(13) into (9)–(10), respectively, to get

$$S_K = -\tilde{k}'''(0) = \text{constant},$$
$$G = \tilde{f}^{\text{iv}}(0) = \text{constant},$$

where a prime denotes a derivative with respect to the variable $\eta \equiv r/\lambda$. Consequently,

$$S_K = S_{K_0}, \quad G = G_0, \tag{18}$$

where the notation $(\cdot)_0$ denotes the initial value. The substitution of (18) into (6) and (8) then yields the transport equations

$$\dot{K} = -\epsilon, \tag{19}$$

$$\dot{\epsilon} = \frac{7}{3\sqrt{15}} S_{K_0} R_t^{\frac{1}{2}} \frac{\epsilon^2}{K} - \frac{7}{15} G_0 \frac{\epsilon^2}{K} \tag{20}$$

for complete self-preserving isotropic turbulence. This is a closed system for the determination of K and ϵ once K_0, ϵ_0, S_{K_0} and G_0 are provided. To simplify the subsequent presentation, the quantity G_0 – which is the coefficient of the term for the destruction of enstrophy in (20) – will henceforth be referred to as the 'palinstrophy coefficient' following the terminology used by Lesieur (1990). Accordingly, the assumption of complete self-preservation is seen to lead to closure in the following sense: *if initial conditions for the skewness and the palinstrophy coefficient are provided – in addition to initial conditions for K and ϵ – then the energy decay can be calculated explicitly for all later times.*

For complete self-preserving isotropic turbulence, the Kármán–Howarth equation (5) takes the form

$$\dot{K}\tilde{f} - K\frac{\dot{\lambda}}{\lambda}\eta\frac{d\tilde{f}}{d\eta} - \left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{K^{\frac{3}{2}}}{\lambda}\eta^{-4}\frac{d(\eta^4\tilde{k})}{d\eta} - 2\nu\frac{K}{\lambda^2}\eta^{-4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right) = 0 \tag{21}$$

or, equivalently,

$$10\tilde{f} + 2\eta^{-4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right) + \eta\frac{d\tilde{f}}{d\eta}\left(\tfrac{7}{3}G_0 - 5\right) = R_\lambda\left(\tfrac{7}{6}S_{K_0}\eta\frac{d\tilde{f}}{d\eta} - \eta^{-4}\frac{d(\eta^4\tilde{k})}{d\eta}\right) \tag{22}$$

after replacing $\dot{\lambda}$ using (11), (19) and (20) with $R_\lambda = (\overline{u^2})^{\frac{1}{2}}\lambda/\nu = (20/3)^{\frac{1}{2}}R_t^{\frac{1}{2}}$. Equation (22) will have a solution if $R_\lambda = \text{constant}$ as first noticed by Dryden (1943); this is a $K \sim t^{-1}$ power law decay. However, (22) also has solutions where $R_\lambda = R_\lambda(t)$ when separability is invoked. The separability condition implies that each side of (22) is equal to zero individually, yielding differential equations from which explicit solutions for $\tilde{f}$ and $\tilde{k}$ may be determined depending on the choice of S_{K_0} and G_0. These solutions were first discovered by Sedov (1944) and later compared with experimental data by Korneyev & Sedov (1976). The particular case for which $G_0 = 3$ so that $\tilde{f}$ is Gaussian – which formally corresponds to the final period of decay – was considered in detail by Bernard (1985). We will briefly examine (22) later to establish the consistency of the present results with those of previous studies. However, our analyses will be based on a fixed point analysis and direct numerical integration of (19) and (20). This will allow us to consider small departures from a self-preserving state as will be demonstrated later.

3. Fixed-point analysis and numerical results

In order to carry out a fixed point analysis of (19) and (20), we will combine these equations into a single transport equation for the turbulence Reynolds number R_t. Since

$$\dot{R}_t = \frac{2K\dot{K}}{\nu\epsilon} - \frac{K^2}{\nu\epsilon^2}\dot{\epsilon} \tag{23}$$

it follows that

$$\dot{R}_t = -\frac{2K}{\nu} - \frac{7}{3\sqrt{15}} S_{K_0} R_t^{\frac{1}{2}} \frac{K}{\nu} + \frac{7}{15} G_0 \frac{K}{\nu}. \tag{24}$$

If the transformed dimensionless time τ – defined by the relation $d\tau = (\epsilon/K)\,dt$ – is introduced into (24), we obtain the equation

$$\frac{dR_t}{d\tau} = R_t\left(\frac{7}{15}G_0 - 2 - \frac{7}{3\sqrt{15}} S_{K_0} R_t^{\frac{1}{2}}\right). \tag{25}$$

The fixed points of (25) are obtained by setting $dR_t/d\tau = 0$ which yields the equation

$$R_{t_\infty}\left(\frac{7}{15}G_0 - 2 - \frac{7}{3\sqrt{15}} S_{K_0} R_{t_\infty}^{\frac{1}{2}}\right) = 0, \tag{26}$$

where $(\cdot)_\infty$ denotes the equilibrium value in the limit as $\tau \rightarrow \infty$. Equation (26) has the solutions

$$R_{t_\infty} = 0 \tag{27}$$

for $\frac{7}{15}G_0 \leqslant 2$, and

$$R_{t_\infty} = \left(\frac{\frac{7}{15}G_0 - 2}{7S_{K_0}/3\sqrt{15}}\right)^2 \tag{28}$$

for $\frac{7}{15}G_0 > 2$. It is a simple matter to show that the fixed points (27) and (28) are *stable nodes* that attract all initial conditions K_0 and ϵ_0. It is also evident from (28) that in order to have an equilibrium high-Reynolds-number isotropic flow field under self-preserving conditions it is necessary that $G_0 \sim R_{t_\infty}^{\frac{1}{2}}$. (By a high-Reynolds-number isotropic turbulence we mean the case where $R_t \gg 1$; for a low-Reynolds-number isotropic turbulence, $R_t = O(1)$.)

By a direct substitution of the fixed point $R_{t_\infty} = 0$ into (20), it can be seen that this fixed point is associated with asymptotic solutions of the differential equations

$$\dot{K} = -\epsilon, \tag{29}$$

$$\dot{\epsilon} = -\tfrac{7}{15}G_0 \frac{\epsilon^2}{K}. \tag{30}$$

It is a simple matter to show that (29) and (30) yield an exact asymptotic solution for K and ϵ of the form

$$K \sim t^{-\alpha}, \quad \epsilon \sim t^{-\alpha-1}, \tag{31}$$

where $\alpha = 1/(\frac{7}{15}G_0 - 1) \geqslant 1$. An exact solution can also be obtained for R_t as a function of the transformed time τ. This solution is given by

$$R_t = R_{t_0}\left[\frac{C^*\exp(\frac{1}{2}C^*\tau)}{C^* - (7/3\sqrt{15})S_{K_0}R_{t_0}^{\frac{1}{2}}\{1 - \exp(\frac{1}{2}C^*\tau)\}}\right]^2 \tag{32}$$

for non-zero values of $C^* \equiv \frac{7}{15}G_0 - 2$, and by

$$R_t = R_{t_0}\left[\frac{1}{1+(7/6\sqrt{15})\,S_{K_0}R_{t_0}^{\frac{1}{2}}\tau}\right]^2 \tag{33}$$

for $C^* = 0$. Since it can easily be shown that

$$\tau = \ln\left(\frac{K_0}{K}\right), \tag{34}$$

it follows from (33) that for $\frac{7}{15}G_0 = 2$ (and $R_{t_0} \gg 1$), we will have $R_t \approx 0$ only when $\tau \gg 1$; this corresponds to the limit of vanishingly small K as a result of (34). From (32), the same conclusion can be drawn when $\frac{7}{15}G_0$ is in its physically realizable range of less than 2 (i.e. for $1.4 \leqslant \frac{7}{15}G_0 \leqslant 2$). It is thus clear that the fixed point $R_{t_\infty} = 0$ can only be achieved in the limit as $K \to 0$, and $t \to \infty$. As we will soon see, this implies physically that an asymptotic power law decay where $K \sim t^{-\alpha}$ (with $\alpha > 1$) is only formally consistent with the *final period of decay* – a fact that will be borne out in subsequent computations.

During the final period of decay there is considerable evidence (Batchelor & Townsend 1948*a*) indicating that $\tilde{f}(\eta) = \exp(-\frac{1}{2}\eta^2)$ (i.e. that $\tilde{f}$ is a Gaussian) in which case (10) implies that $G_0 = 3$ and, consequently, that $\alpha = \frac{5}{2}$. The same result is also reached by assuming constancy of the Loitsianskii integral

$$\overline{u^2}\int_0^\infty r^4 f(r,t)\,\mathrm{d}r = \text{constant} \tag{35}$$

during decay, which appears to be an acceptable assumption for the final period. In particular, from (12) it follows that (35) is equivalent to (cf. Hinze 1975)

$$\overline{u^2}\lambda^5\int_0^\infty \eta^4\tilde{f}(\eta)\,\mathrm{d}\eta = \text{constant} \tag{36}$$

so that (11) and (36) imply that

$$\frac{K^{\frac{7}{2}}}{\epsilon^{\frac{5}{2}}} = \text{constant}. \tag{37}$$

When (37) is combined with (31) it follows that $\alpha = 5/2$ – the celebrated Batchelor (1948) result.

Now we will show that the non-zero fixed point (28) is consistent with high-Reynolds-number isotropic turbulence. The substitution of the fixed point (28) into (19) and (20) yields the differential equations

$$\dot{K} = -\epsilon, \tag{38}$$

$$\dot{\epsilon} = -2\frac{\epsilon^2}{K}, \tag{39}$$

which have the asymptotic solution

$$K \sim t^{-1} \tag{40}$$

$$\epsilon \sim t^{-2} \tag{41}$$

(name, a t^{-1} power law decay for the turbulent kinetic energy). From (32), it is clear that this non-zero fixed point is approached rapidly for sufficiently large G_0. More precisely, for $G_0 = O(10)$, a t^{-1} asymptotic decay law will be established within a few eddy turnover times – a feature that will be demonstrated in later computations.

Since it can be shown that (see Batchelor & Townsend 1948*b*)

$$G = \frac{30\nu}{7} \frac{\overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}} \Big/ \overline{\omega_k \omega_k}}{\epsilon/K} \tag{42}$$

it follows that G is a ratio of turbulent to dissipative timescales. It is a simple matter to prove that (see Hinze 1975)

$$\frac{\nu \overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}}{\overline{\omega_k \omega_k}} \sim \frac{\nu \int_0^\infty \kappa^4 E(\kappa, t)\, \mathrm{d}\kappa}{\int_0^\infty \kappa^2 E(\kappa t)\, \mathrm{d}\kappa}, \tag{43}$$

where $E(\kappa, t)$ is the energy spectrum and κ is the wavenumber. For complete self-preservation, $E(\kappa, t)$ scales with the Taylor microscale, from which it follows that

$$\frac{\nu \overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}}{\overline{\omega_k \omega_k}} \sim \frac{\nu}{\lambda^2} \sim \frac{\epsilon}{K} \tag{44}$$

and, hence, that

$$G \sim \text{constant} \tag{45}$$

as shown earlier. However, (43) is a correlation dominated by the high wavenumbers (i.e. small scales) and it would therefore seem more reasonable that $E(\kappa, t)$ should scale with the Kolmogorov lengthscale $l_K \equiv \nu^{\frac{3}{4}}/\epsilon^{\frac{1}{4}}$. If this is the case, then

$$\frac{\nu \overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}}{\overline{\omega_k \omega_k}} \sim \frac{\nu}{l_K^2} \tag{46}$$

and

$$G \sim \frac{\nu}{l_K^2}\frac{K}{\epsilon} \sim R_t^{\frac{1}{2}}. \tag{47}$$

Equations (45) and (47) appear to be contradictory; however, they are not, since in a $K \sim t^{-1}$ power law decay

$$R_t = \text{constant}. \tag{48}$$

Furthermore, $R_t = \text{constant}$ resolves the imbalance between the two terms on the right-hand side of the dissipation rate equation (20) since the first term (i.e. the vortex stretching) is initially of $O(R_t^{\frac{1}{2}})$ while the second term (i.e. the viscous destruction) is of $O(1)$. This leads us to the following physical interpretation: *a $K \sim t^{-1}$ power law decay is the asymptotic state toward which a self-preserving isotropic turbulence is driven at high Reynolds numbers in order to resolve the fundamental imbalance between vortex stretching and viscous diffusion. In the process of resolving this imbalance, compatibility with Kolmogorov scaling is achieved for the small-scale correlations.* Since this consistency with Kolmogorov scaling – which, on physical grounds should be satisfied at high Reynolds numbers – is achieved when $\frac{7}{15}G_0 \sim R_{t_\infty}^{\frac{1}{2}}$ which is greater than 2, it is clear that the non-zero fixed point (28) is the physically consistent asymptotic solution for high-Reynolds-number self-preserving flows.

We will now examine numerical solutions of (19) and (20) for complete self-preservation. With the exception of the recent work of Bernard (1985), we have not seen detailed numerical results published on the decay of K and ϵ in complete self-preserving isotropic turbulence. An examination of these results will amplify the

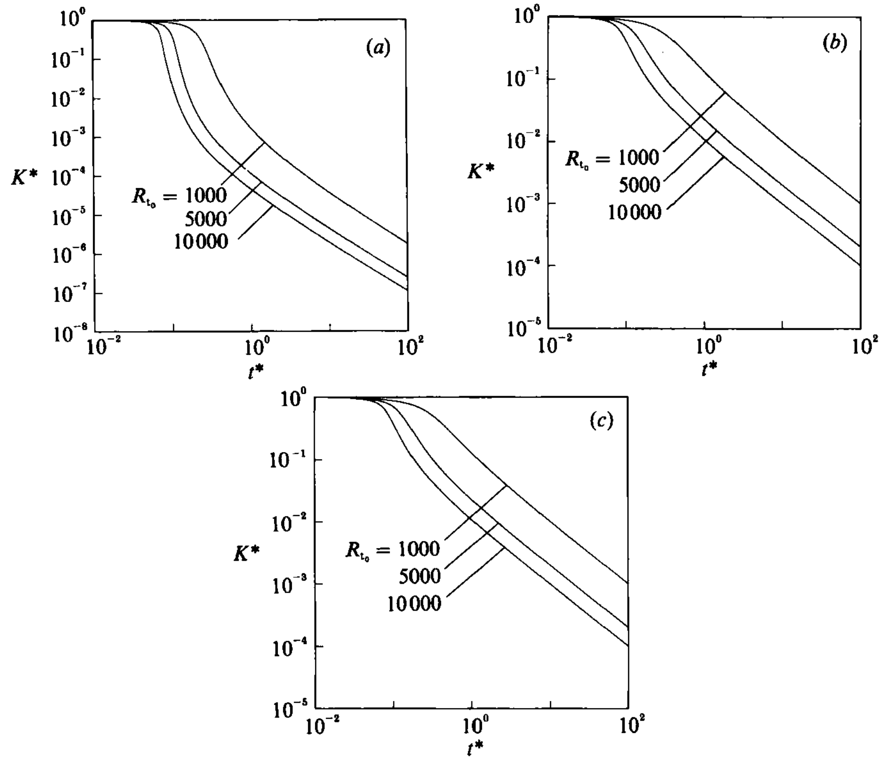

FIGURE 1. Decay of turbulent kinetic energy in complete self-preserving isotropic turbulence for initial turbulence Reynolds numbers $R_{t_0} = 1000$, 5000 and 10000: (a) $C_{\epsilon_2} = 1.92$, (b) $C_{\epsilon_2} = 5$, and (c) $C_{\epsilon_2} = 8$.

points discussed in this section and will shed some interesting new light on how the self-preservation assumption compares with experiments. In figures 1(a)–1(c), the decay of the turbulent kinetic energy is shown for three initial turbulence Reynolds numbers ($R_{t_0} = 1000$, 5000 and 10000) and three different initial conditions for G (i.e. $C_{\epsilon_2} = 1.92$, 5.0, and 8.0 where $C_{\epsilon_2} = \frac{7}{15}G_0$). For these calculations, as well as the ones to follow, $K^* \equiv K/K_0$, $t^* \equiv \epsilon_0 t/K_0$ and the skewness

$$S_{K_0} = 0.5, \tag{49}$$

which is in close proximity to the values obtained from physical experiments in this range of Reynolds numbers. From these figures it is clear that the self-preserving solution has an initial transient where the kinetic energy is fairly flat; then the kinetic energy begins to asymptote from above to a power law decay as evidenced by a straight line on these logarithmic plots. Two conclusions can be drawn from these results. First, for $C_{\epsilon_2} \lesssim 2$ and $R_{t_0} \gg 1$, the kinetic energy does not asymptote to a $t^{-\alpha}$ power law decay until after an extremely large number of eddy turnover times by which time the turbulence has decayed to a tiny fraction of its initial intensity. Second, for C_{ϵ_2} sufficiently larger than 2, the kinetic energy asymptotes to a t^{-1} power law decay within a few eddy turnover times; however unless $C_{\epsilon_2} \sim R_{t_0}^{\frac{1}{2}}$, consistent with Kolmogorov scaling, the turbulence intensity will drop precipitously before this asymptotic state is achieved.

To further illustrate these points, the computed turbulent kinetic energy is compared with its corresponding asymptotic power law decay for increasing values of C_{ϵ_2} in figures 2(a)–2(c) which are for initial turbulence Reynolds numbers of 1000,

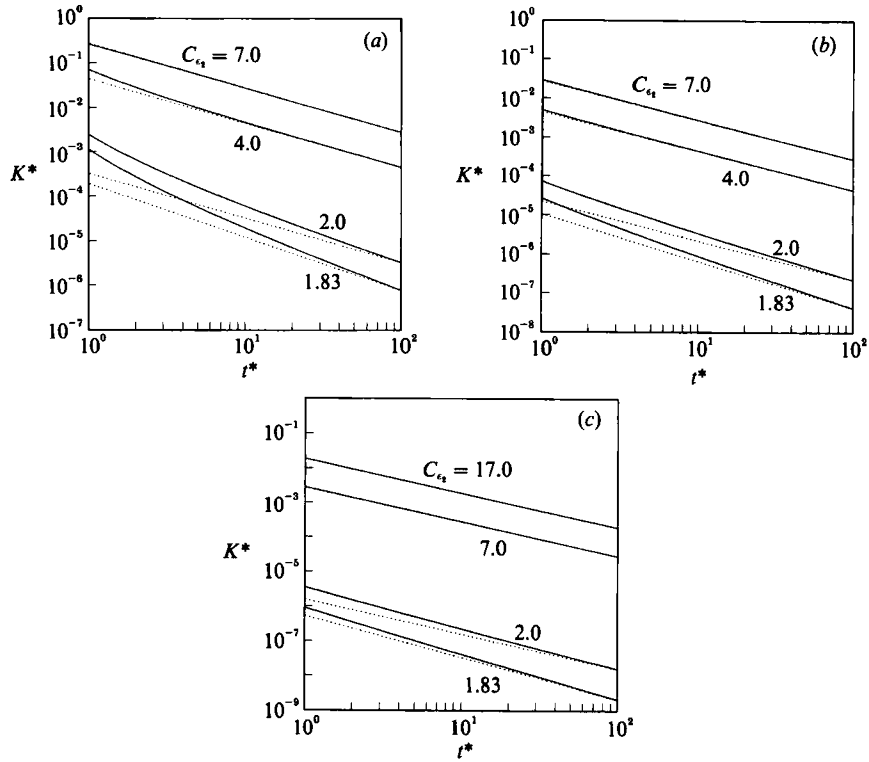

FIGURE 2. Decay of turbulent kinetic energy for a variety of initial conditions on G ($C_{\epsilon_2} = \frac{7}{15}G_0$): —, self-preserving solution; ---, asymptotic solution $K \sim t^{-\alpha}$ where $\alpha = (C_{\epsilon_2}-1)^{-1}$ for $C_{\epsilon_2} < 2$ and $\alpha = 1$ for $C_{\epsilon_2} \geqslant 2$. (a) $R_{t_0} = 1000$, (b) $R_{t_0} = 10000$ and (c) $R_{t_0} = 100000$.

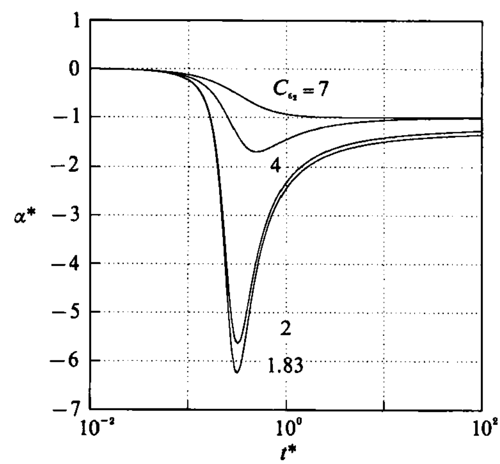

FIGURE 3. Time evolution of the exponent of the decay law for a variety of initial conditions on G ($C_{\epsilon_2} = \frac{7}{15}G_0$).

10000 and 100000, respectively. It is clear from these figures that for $C_{\epsilon_2} \leqslant 2$, the turbulent kinetic energy does not reach its asymptotic power law decay even after 100 eddy turnover times! However, for C_{ϵ_2} sufficiently larger than 2, the turbulent kinetic energy asymptotes to a t^{-1} power law decay within a few eddy turnover times. This can be seen even more vividly in figure 3 where $\alpha^* \equiv \mathrm{d}(\log K^*)/\mathrm{d}(\log t^*)$ is plotted as a function of $\log t^*$ for $R_{t_0} = 1000$. If there is an asymptotic power law

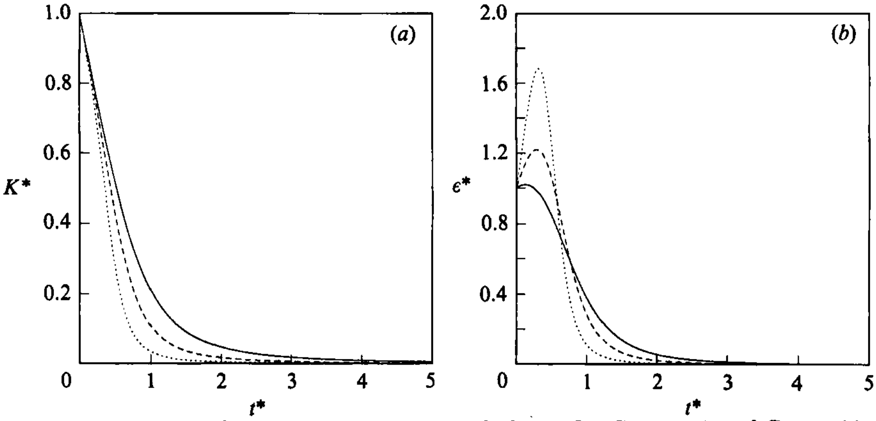

FIGURE 4. Decay of self-preserving isotropic turbulence for $C_{\epsilon_2} = 1.83$ and $R_{t_0} = 100$ (—), 200 (– –) and 400 (- - -): (*a*) turbulent kinetic energy, and (*b*) turbulent dissipation rate.

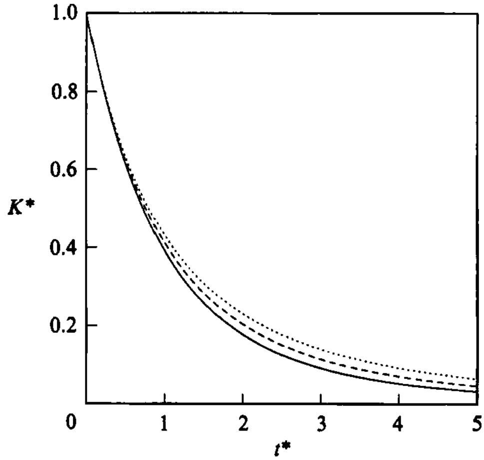

FIGURE 5. Self-preserving isotropic turbulence at low turbulence Reynolds numbers ($C_{\epsilon_2} = 1.4$): —, $R_{t_0} = 1$; – –, $R_{t_0} = 0.25$; - - -, equation (31).

decay, this derivative will asymptote to the exponent of the decay law. It is clear that for $C_{\epsilon_2} = 7$, an exponent of 1 is approached quickly; however, for $C_{\epsilon_2} = 1.83$ and 1.92 (initial conditions which ultimately yield a power law decay with an exponent of approximately 1.2 and 1.1, respectively) an asymptotic state is not achieved even after 100 eddy turnover times. Furthermore, for $C_{\epsilon_2} < 2$ and for large initial turbulence Reynolds numbers $R_{t_0} \gg 1$, there is a precipitous drop in the turbulent kinetic energy before a power law decay is achieved; this is due to the early transient when vortex stretching causes a considerable rise in the dissipation (see figures 4*a*–4*b*).

Since the self-preserving solutions for $C_{\epsilon_2} < 2$ only asymptote to a power law decay in the limit as $K \to 0$ and $t \to \infty$, it is reasonable to associate them exclusively with the final period of decay. Experiments tend to indicate that the final period of decay is entered for $R_t < 1$ wherein the exponent of the decay is approximately 2.5 (cf. Hinze 1975). As noted earlier, this decay law is obtained asymptotically for self-preserving isotropic turbulence if $C_{\epsilon_2} = 1.4$ – a result obtained by invoking Loitsianskii's invariant. In figure 5, the decay of the turbulent kinetic energy when $C_{\epsilon_2} = 1.4$ is shown for the initial turbulence Reynolds numbers $R_{t_0} = 0.25$ and 1.0. It is clear from

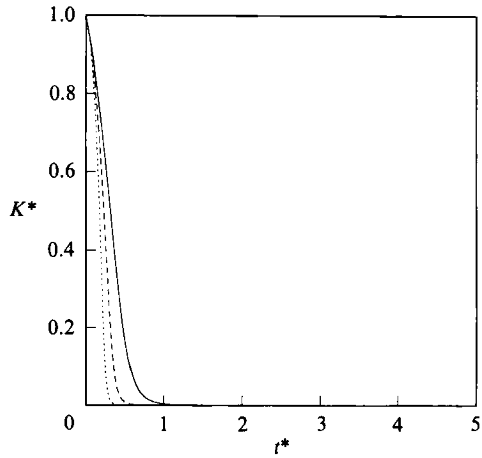

FIGURE 6. Decay of turbulent kinetic energy in self-preserving isotropic turbulence for $C_{\epsilon_2} = 1.4$: —, $R_{t_0} = 200$; ––, $R_{t_0} = 500$; ---, $R_{t_0} = 1000$.

these results that for $R_{t_0} < 1$ the solutions begin to asymptote to the power law decay (31) with an exponent $\alpha = 2.5$. Since C_{ϵ_2} must equal 1.4 during the final period of decay – and, since C_{ϵ_2} is a constant for self-preserving isotropic turbulence – it follows that *the entire decay process from high-Reynolds-number initial conditions to the final period of decay cannot be described within the framework of complete self-preservation.* This conclusion results from the fact that the only consistent self-preserving solution at high initial Reynolds numbers yields a $K \sim t^{-1}$ power law decay wherein R_t asymptotes to a constant – a state of affairs that precludes the description of the later stages of decay. Furthermore, the value of $C_{\epsilon_2} = 1.4$, which describes the final period of decay, yields unphysical results for the early stages of a high Reynolds number isotropic turbulence (i.e. it predicts an early time transient where there is a precipitous drop in the turbulent kinetic energy; see figure 6). In order to describe the entire decay process of a high-Reynolds-number isotropic turbulence, G as well as S_K must vary with time – a possibility that is precluded by the assumption of complete self-preservation which renders them constant.

4. Comparisons with alternative theoretical analyses and experiments

The results derived in the previous section are consistent with those of Batchelor (1948) for low turbulence Reynolds numbers; however, our high-Reynolds-number asymptotic solution yields $K \sim t^{-1}$ whereas in Batchelor's solution $K \sim t^{-\frac{10}{7}}$. The reason for this difference is simple: as alluded to earlier, Batchelor also found the $K \sim t^{-1}$ solution but dismissed it as a viable result since Loitsianskii's integral was not a dynamic invariant therein. Interestingly enough, an earlier experimental study by Batchelor & Townsend (1947) yielded results that were far more suggestive of a $K \sim t^{-1}$ rather than a $K \sim t^{-\frac{10}{7}}$ power law decay. Despite the fact that Batchelor (1948) states that the $K \sim t^{-\frac{10}{7}}$ decay law is a complete self-preserving solution, in reality it is only a partial self-preserving solution since it corresponds to the inviscid Kármán–Howarth equation (see Hinze 1975). It is our view that since the $K \sim t^{-1}$ asymptotic decay law is a formal solution to the full Kármán–Howarth equation, it should not be casually dismissed unless it is in incontrovertible contradiction of experiments or other exact theoretical results.

As mentioned earlier, Dryden (1943) postulated a $K \sim t^{-1}$ power law decay based on a direct analysis of the Kármán–Howarth equation. He observed – as is evident from (22) – that the Kármán–Howarth equation will allow for self-similar solutions if

$$R_\lambda = \text{constant}, \tag{50}$$

which yields $K \sim t^{-1}$ as a direct consequence of (6). However, there are other temporally varying solutions to (22); complete self-preservation only requires that R_λ asymptote to a constant. Sedov (1944) studied solutions of (22) obtained by applying the separability constraint

$$10\tilde{f} + 2\eta^{-4}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^4 \frac{\mathrm{d}\tilde{f}}{\mathrm{d}\eta}\right) + \eta \frac{\mathrm{d}\tilde{f}}{\mathrm{d}\eta}\left(\tfrac{7}{3}G_0 - 5\right) = 0, \tag{51}$$

$$\left(\tfrac{7}{6}S_{K_0}\eta\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\eta} - \eta^{-4}\frac{\mathrm{d}(\eta^4\tilde{k})}{\mathrm{d}\eta}\right) = 0, \tag{52}$$

which renders $R_\lambda = R_\lambda(t)$, consistent with (19) and (20). Solutions to (51) and (52) have not been studied in great depth subsequent to Sedov (1944) who showed that $\tilde{f}(\eta) = M(5/2\gamma, 5/2, -\gamma\eta^2/2)$ where M is the confluent hypergeometric function and $\gamma = \frac{7}{6}G_0 - \frac{5}{2}$. Batchelor (1948) expressed concern over the fact that this solution leads to a unique determination of both $\tilde{f}$ and $\tilde{k}$; however, although he suspected that the Sedov solution was unphysical, he stated that he was 'not able to find any definite anomalies'. In the limit as $R_\lambda \to 0$ it is clear that (51) is a direct consequence of the Kármán–Howarth equation. Consequently, it is not surprising that the Sedov solution for the final period of decay yields physically interesting solutions as recently demonstrated by Bernard (1985). However, it will now be shown definitively that the Sedov solution yields unphysical results at high turbulence Reynolds numbers. In figure 7, the results of a numerical solution of (51) for $\tilde{f}$ are shown for a variety of values of G_0 ranging from 3 to 60. For $G_0 = 3$ it can be shown analytically that $\tilde{f} = \exp(-\frac{1}{2}\eta^2)$, yielding an energy spectrum of the form

$$E^*(\kappa^*) = \frac{1}{(2\pi)^{\frac{1}{2}}}\kappa^{*4}\exp(-\tfrac{1}{2}\kappa^{*2}), \tag{53}$$

where $\kappa^* = \kappa\lambda$ and $E^* = E/\overline{u^2}\lambda$. Equation (53) is obtained from the identity

$$E^*(\kappa^*) = \frac{1}{\pi}\int_0^\infty \tilde{f}(\eta)\,(\kappa^*\eta\sin\kappa^*\eta - \kappa^{*2}\eta^2\cos\kappa^*\eta)\,\mathrm{d}\eta \tag{54}$$

(cf. Batchelor 1953). This result – which has $E^*(\kappa^*) \sim \kappa^{*4}$ at low wavenumbers and has $E^*(\kappa^*)$ decaying exponentially at high wavenumbers – is consistent with established results on the final period of decay (cf. Hinze 1975). However, for sufficiently large G_0, it is a simple matter to show from (51) that

$$\tilde{f}(\eta) \sim \eta^{-5/\gamma} \tag{55}$$

for $\eta \gg 1$. This explains why $\tilde{f}(\eta)$ is so slow to asymptote to zero when $G_0 > 10$ in figure 7. In fact for

$$\tfrac{7}{15}G_0 > 2$$

it follows from (54) and (55) that $E^*(\kappa^*)$ becomes singular. From (28) it can then be concluded that the Sedov solution will yield a singular energy spectrum when

$$R_{t_\infty} > 0.$$

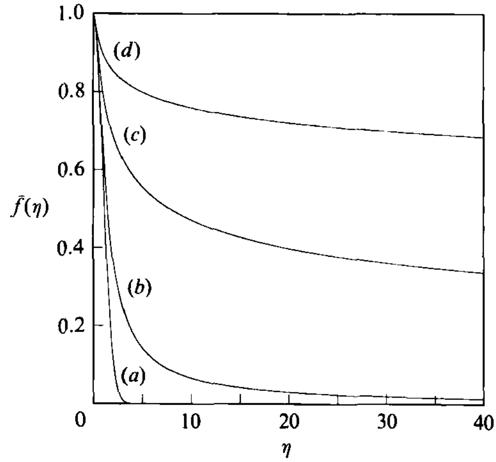

FIGURE 7. Sedov solution for the two-point double longitudinal velocity correlation: (a) $G_0 = 3$, (b) $G_0 = 6$, (c) $G_0 = 20$, and (d) $G_0 = 60$.

It is thus clear that the Sedov solution yields untenable results, at high Reynolds numbers, for the double and triple two-point longitudinal velocity correlations.

The major deficiency with the approaches of Dryden and Batchelor – as well as that of Sedov – lies in the use of the *self-similar* Kármán–Howarth equation based on the assumptions (12) and (13), which does not allow for the treatment of small departures from a state of complete self-preservation. Such small departures can be characterized by the perturbations

$$S_K = S_{K_0} + \delta S_K(t), \tag{56}$$

$$G = G_0 + \delta G(t), \tag{57}$$

where $\|\delta S_K\|/S_{K_0} \ll 1$, $\|\delta G\|/G_0 \ll 1$ and $\delta S_K(t)$, $\delta G(t) \rightarrow 0$ as $t \rightarrow \infty$. The substitution of (56) and (57) into (6) and (8) yields the governing equations for small departures from a state of self-preservation. If we denote by δK and $\delta\epsilon$ the departures from the self-preserving solutions K and ϵ obtained from (19) and (20), it follows that for the perturbations (56) and (57) we will have

$$\frac{\|\delta K\|}{\|K\|} \ll 1, \quad \frac{\|\delta \epsilon\|}{\|\epsilon\|} \ll 1$$

since (19) and (20) have fixed points that are stable nodes (cf. Guckenheimer & Holmes 1986). Consequently, (19) and (20) will yield an excellent approximation for isotropic decay when there are extremely small departures from a state of complete self-preservation. In contrast to this nice behaviour, the Kármán–Howarth equation becomes indeterminate when subjected to infinitesimal perturbations from a self-preserving state. Hence, it appears that the one-point equations (19) and (20) form a broader basis for the analysis of the energy decay of self-preserving isotropic decay than does (22).

The general solution to the complete self-preserving isotropic decay equations (19) and (20) at high Reynolds numbers is shown schematically in figure 8 (this is for the physically significant case where $R_{t_0} > R_{t_\infty}$ so that the turbulence Reynolds number decays). There is an early time transient (region AB) where the turbulent kinetic energy is flat; it is eventually followed by the asymptotic region CD where $K \sim t^{-1}$.

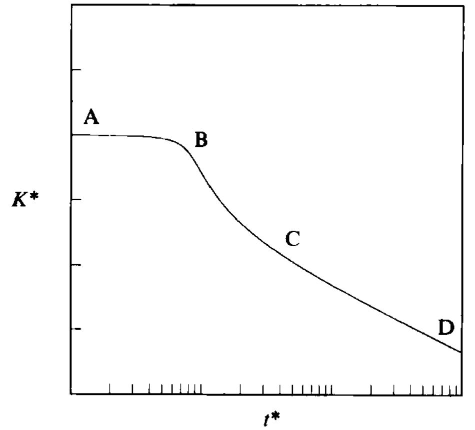

FIGURE 8. Schematic of the complete self-preserving solution for the decay of turbulent kinetic energy at high Reynolds numbers ($R_{t_0} > R_{t_\infty}$).

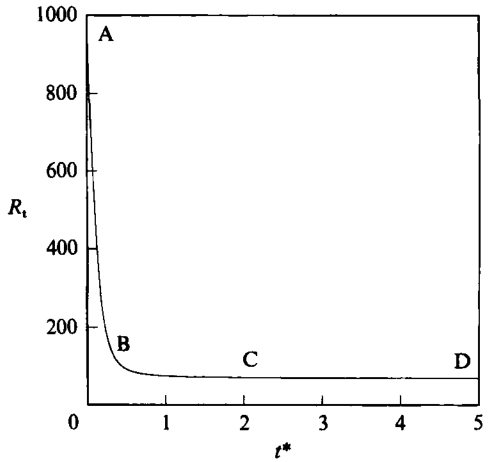

FIGURE 9. Decay of the turbulence Reynolds number in complete self-preserving isotropic turbulence: $R_{t_0} = 1000$, $C_{\epsilon_2} = 4.5$.

These two regions are connected by the overlap region BC. The initial transient AB evolves on the Kolmogorov timescale $(\nu/\epsilon)^{\frac{1}{2}}$ during which time there is a precipitous drop in the turbulence Reynolds number (see figure 9). On the other hand, the overlap region BC evolves on the turbulence timescale K/ϵ; in this region the turbulence Reynolds number R_t becomes close to R_{t_∞}, approaching it asymptotically from above. As a direct consequence of the perturbation analysis discussed above, the overlap region BC can be set into strong approximate agreement with the asymptotic approach to a state of complete self-preservation. These results have a direct bearing on how the complete self-preserving solution compares with physical experiments, as we will soon see.

It is widely believed that a $K \sim t^{-1}$ asymptotic decay law is in violation of experimental data for isotropic turbulence. These experimental data (see Uberoi 1963; Kistler & Vrebalovich 1966; Comte-Bellot & Corrsin 1966, 1971; Warhaft & Lumley 1978 and Mohamed & La Rue 1990) have yielded power law decays with exponents varying from 1 to 1.4 with a mean of approximately 1.25. However, great caution must be exercised in using these data to dismiss the possibility of a $K \sim t^{-1}$

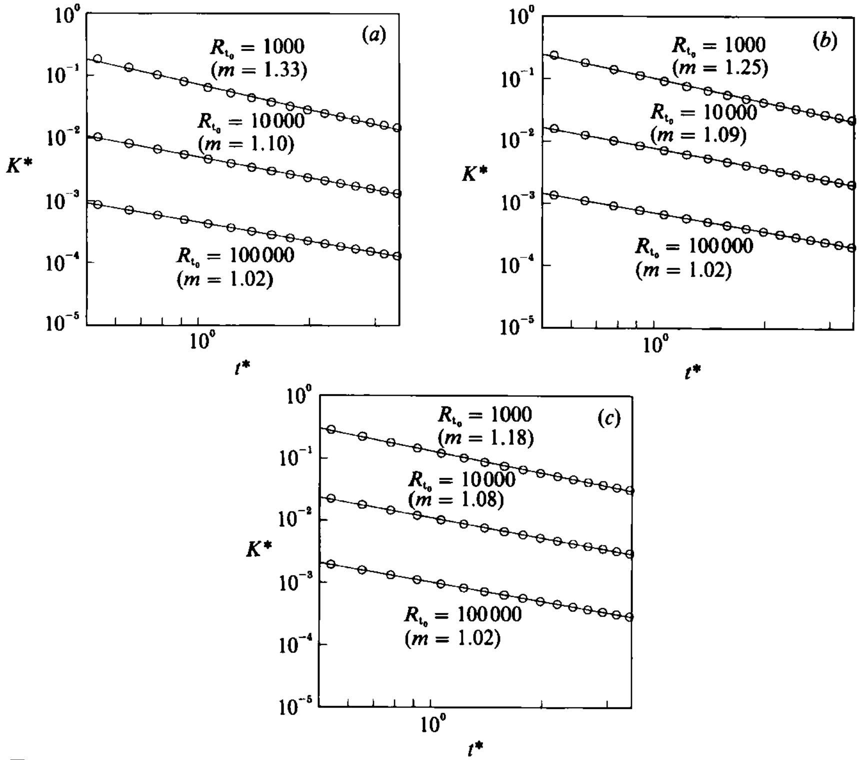

FIGURE 10. Decay of turbulent kinetic energy: o, self-preserving solution; —, $K \propto t^{-m}$. (a) $C_{\epsilon_2} = 4$, (b) $C_{\epsilon_2} = 4.5$, and (c) $C_{\epsilon_2} = 5$.

asymptotic power law decay at high Reynolds numbers since most of these data are for a limited number of eddy turnover times (typically for $\epsilon_0 t/K_0 < 4$). If the self-preserving solution is examined for this same limited number of eddy turnover times it follows that the resulting solution can be fitted to an excellent degree of approximation by a power law decay with exponents in the range of 1 to 1.4 depending on the initial conditions; the lower the Reynolds number, the longer the solution takes to reach an asymptotic state and the larger the exponent is during the early stages of decay (see figure 10*a*–*c*). Consequently, if one examined in isolation the self-preserving solutions for the first few eddy turnover times (with the short early time transient omitted), one could erroneously conclude that there was an asymptotic power law decay with an exponent in the range of 1 to 1.4 depending on the initial conditions; in reality, all of these solutions are asymptoting to a t^{-1} power law decay. The solutions shown in figure 10(*a*–*c*) correspond to the overlap region BC shown in figure 8 and, hence, can be associated with the asymptotic approach to a state of complete self-preservation. An argument has been raised recently by Walker & Corrsin (1985) and Walker (1986) that the physical experiments may not go far enough to see a t^{-1} power law decay. Unless the initial turbulence Reynolds number is extremely large, an asymptotic state may not be achieved in the first few eddy turnover times. In this regard it is interesting to note that the only extremely high-Reynolds-number experiment (i.e. Kistler & Vrebalovich 1966) and large-eddy turnover time experiment (Walker 1986) did measure a $K \sim t^{-1}$ asymptotic power law decay. Consequently, existing experiments cannot rule out the possibility of a $K \sim t^{-1}$ asymptotic power law decay at high Reynolds numbers and do not warrant

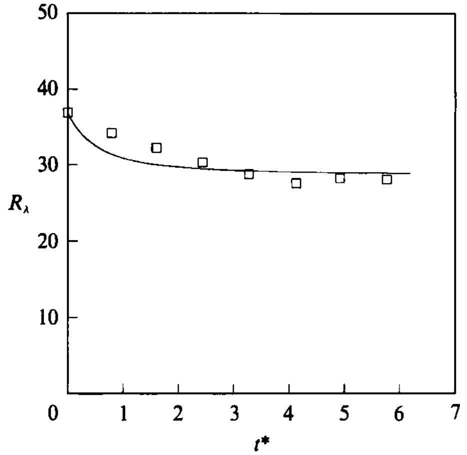

FIGURE 11. Time evolution of the Taylor microscale Reynolds number in isotropic decay ($G_0 \approx 9.5$, $S_{K_0} \approx 0.4$ and $R_{\lambda_0} \approx 37$): —, complete self-preserving solution; □, experiments of Batchelor & Townsend (1947).

the dismissal of the complete self-preserving solution discussed herein. Furthermore, Rosen (1981) recently derived a t^{-1} asymptotic power law decay based on alternative ideas from statistical mechanics.

Some comments are in order concerning how the complete self-preserving solution compares quantitatively with existing experimental data for the first few eddy turnover times. This corresponds to the overlap region of the self-preserving solution which can be fitted with an approximate power law decay with exponents in the range of 1.0–1.4. Here, the larger exponents correspond to smaller initial turbulence Reynolds numbers R_{t_0} whereas exponents close to one correspond to large values of R_{t_0}. Consequently, in the overlap region – which is of a duration as long as that in most of the physical experiments – an approximate $K \sim t^{-1}$ power law decay occurs in the limit of infinite Reynolds numbers (i.e. as $R_{t_0} \rightarrow \infty$). This result is completely consistent with the results of Ling & Wan (1972) and Barenblatt & Gavrilov (1974) as well as with earlier speculations by Lin (1948) who linked $K \sim t^{-1}$ with the infinite-Reynolds-number limit. At lower turbulence Reynolds numbers (i.e. $R_{t_0} \leqslant 1000$), larger decay exponents in the range of 1.25–1.4 are obtained numerically as shown in figure 10. This is generally compatible with the most recent experiments of Mohamed & La Rue (1990) for $100 \leqslant R_{t_0} \leqslant 250$ who found a decay exponent of 1.3 which they claimed was independent of the initial conditions provided that there was a proper choice of the virtual origin. Unfortunately, detailed quantitative comparisons with experiments for the energy decay are not possible owing to the difficulty in measuring G. Since $G \propto \int_0^\infty \kappa^4 E(\kappa, t)\,\mathrm{d}\kappa$, measurements are required at very high wavenumbers in order to reliably determine G; none of the existing experiments measure $E(\kappa, t)$ at sufficiently high wavenumbers (e.g. in the Comte-Bellot & Corrsin 1971 experiment, $\kappa^4 E(\kappa, 0)$ is still an increasing function of κ at the highest wavenumber measured). We will nonetheless make a direct comparison with the experiment of Batchelor & Townsend (1947) since they provided explicit estimates of G from their measurements. In figure 11, the time evolution of R_λ predicted by the complete self-preserving theory is compared with the measurements of Batchelor & Townsend (1947) for, $R_{\lambda_0} \approx 37$, $G_0 \approx 9.5$ and $S_{K_0} \approx 0.4$. It can be seen that the theory is well within the range of the experimental data. However, any conclusions drawn from this must be guarded owing to the age of these experiments and the uncertainty in measuring G – a deficiency in all existing data that must be overcome before detailed quantitative comparisons between theory and experiments can be made reliably.

Now, we will address the interesting controversy generated recently by George (1987, 1989, 1992). He claimed to find self-preserving solutions – with the Taylor microscale as the similarity lengthscale – which exist for *all* turbulence Reynolds numbers. These solutions were characterized by an asymptotic power law decay where the exponent is determined by the initial conditions. George arrived at this alternative self-preserving solution by relaxing the classical similarity constraint (13). He argued that the normalization of the two-point triple velocity correlation

$$T(r,t) = \overline{u^2(\boldsymbol{x},t)\,u(\boldsymbol{x}+\boldsymbol{r},t)} \tag{58}$$

by $(\overline{u^2})^{\frac{3}{2}}$ to form $k(r,t)$ is arbitrary since its one-point contraction $T(0,t) \equiv \overline{u^3}$ is zero. (This stands in contrast to the formulation of $f(r,t)$ which is obtained by normalizing the two-point double velocity correlation with its one-point contraction $\overline{u^2}$). Consequently, George argued that constraint (13) should be replaced with the alternative constraint

$$\frac{T(r,t)}{w(t)} = \tilde{T}\left(\frac{r}{\lambda}\right), \tag{59}$$

where $w(t)$ is a suitable weighting function. Then – from the definition of S_K in (9) and the Kármán–Howarth equation (22) – instead of the constraints

$$R_\lambda = \text{constant}, \quad S_K = \text{constant} \tag{60}$$

which render a t^{-1} power law decay, we get the constraints

$$R_\lambda S_K = \text{constant}, \quad R_\lambda w K^{-\frac{3}{2}} = \text{constant} \tag{61}$$

which allow for the possibility of an alternative decay law within the general framework of self-preservation. From (61) it follows that

$$S_K \propto R_t^{-\frac{1}{2}} \tag{62}$$

and that

$$w \propto K^{\frac{3}{2}} R_t^{-\frac{1}{2}}, \tag{63}$$

which shows, incidentally, that w cannot be chosen arbitrarily. Since the proportionality constant in (62) must be $S_{K_0} R_{t_0}^{\frac{1}{2}}$ – and since G still remains a constant G_0 during the decay – this alternative self-preservation leads to the decay equations

$$\dot{K} = -\epsilon, \tag{64}$$

$$\dot{\epsilon} = \frac{7}{3\sqrt{15}} S_{K_0} R_{t_0}^{\frac{1}{2}} \frac{\epsilon^2}{K} - \frac{7}{15} G_0 \frac{\epsilon^2}{K} \tag{65}$$

instead of (19) and (20). Equations (64) and (65) yield the closed-form solution for the energy decay

$$K = K_0\left(1 + \frac{1}{\beta}\frac{\epsilon_0 t}{K_0}\right)^{-\beta} \tag{66}$$

where

$$\beta = \left(\frac{7}{15} G_0 - \frac{7}{3\sqrt{15}} S_{K_0} R_{t_0}^{\frac{1}{2}} - 1\right)^{-1}. \tag{67}$$

This is indeed a power law decay with an exponent that depends on the initial conditions as claimed by George (1987, 1989, 1992).

Several observations can be made about this alternative self-preserving solution of George (1987, 1989, 1992). The asymptotic relation for S_K in (62) cannot hold for long times since, if the skewness is to be finite at moderate R_t, according to (62) it must vanish in the limit as R_t goes to infinity – an unacceptable physical result that would imply vanishing transfer in the limit of infinite Reynolds numbers. Hence, (62) can only hold for a few eddy turnover times until the skewness peaks: a result which is in general agreement with experiments as shown recently by George (1992) (also see Van Atta & Antonia 1980). After S_K peaks it can either remain constant for a considerable period of time or it can begin to decrease gradually. If the former occurs, then the George solution will constitute an asymptotic solution for the approach to a state of complete self-preservation where

$$S_K = S_{K_0} + \delta S_K(t), \quad G = G_0.$$

Since

$$S_K = \frac{S_{K_0} R_{t_0}^{\frac{1}{2}}}{R_t^{\frac{1}{2}}} = \frac{S_{K_0} R_{t_0}^{\frac{1}{2}}}{(R_{t_0} - \delta R_t)^{\frac{1}{2}}}$$

it follows that $\delta S_K(t) \approx \frac{1}{2} S_{K_0} \delta R_t(t)/R_{t_0}$ for $\delta R_t / R_{t_0} \ll 1$. Hence, it is now not surprising that the George solution yields a power law decay with an exponent that depends on the initial conditions – the same results obtained for the overlap region of the complete self-preserving solution illustrated in figure 10. By relaxing the classical self-similar constraint for the two-point triple velocity correlation, the George (1987) solution bears some resemblance to the self-preserving solutions of the second-kind discussed by Barenblatt (1979). It appears to be a physically consistent candidate for the asymptotic approach to a state of complete self-preservation.

Finally, a few comments are in order concerning the implications of these results for turbulence modelling. In the commonly used turbulence models, the dissipation rate equation is modelled as

$$\dot{\epsilon} = -C_{\epsilon_2} \frac{\epsilon^2}{K} \tag{68}$$

for isotropic decay, where C_{ϵ_2} is a constant (cf. Launder & Spalding 1974; Speziale 1991). Equation (68) is derived by invoking Kolmogorov scaling for G which requires that

$$G = C_1 R_t^{\frac{1}{2}} + C_2, \tag{69}$$

where C_1 and C_2 are constants; an equilibrium hypothesis is then made wherein $C_1 = (\sqrt{15}/3) S_K$ so that the leading-order part of the destruction of dissipation term annihilates the vortex stretching term in (8) yielding (68). In contrast to (68), the complete self-preserving solution has a dissipation rate transport equation of the general mathematical form

$$\dot{\epsilon} = C_{\epsilon_3} R_t^{\frac{1}{2}} \frac{\epsilon^2}{K} - C_{\epsilon_2} \frac{\epsilon^2}{K}, \tag{70}$$

where C_{ϵ_2} and C_{ϵ_3} are constants. Equation (70) can also be derived based on Kolmogorov scaling (69) when departures from equilibrium are allowed wherein $C_1 \neq (\sqrt{15}/3) S_K$. The addition of the unbalanced vortex stretching term in (70) allows for a better treatment of departures from equilibrium in several ways. First, as shown earlier in figure 10(*a–c*), the self-preservation model allows the description of the initial stages of isotropic decay where the exponent of the decay law can vary

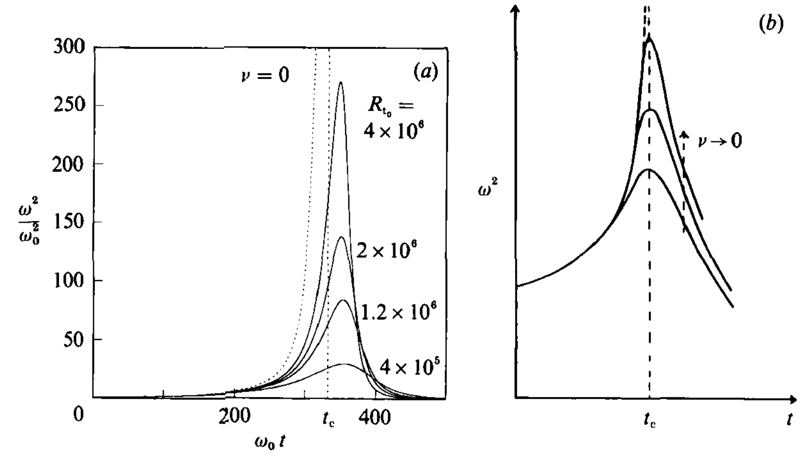

FIGURE 12. Time evolution of the enstrophy at high turbulence Reynolds numbers: (a) complete self-preserving solution ($C_{\epsilon_2} \doteq 2$) and (b) schematic from Lesieur (1990) based on EDQNM.

mildly with the initial conditions – a feature observed in physical experiments. On the other hand, the more commonly used model (68) predicts a universal decay law where $K \sim t^{-1/(C_{\epsilon_2}-1)}$ for all R_{t_0} and all $t > 0$. Second, the self-preserving solution can accommodate the limit of zero viscosity. In this limit, it is a simple matter to show that (20) predicts a finite-time enstrophy blow-up at the critical time

$$t_c = \frac{6\surd 15}{7S_{K_0}\omega_0}, \tag{71}$$

where ω_0^2 is the initial enstrophy. In figure 12(a), the time evolution of the enstrophy corresponding to the complete self-preserving solution is shown for a variety of increasing R_{t_0}; it is clear that an enstrophy crisis is predicted for $R_{t_0} \gg 1$ which eventually leads to a finite-time enstrophy blow-up in the limit as $\nu \to 0$. These results are in excellent qualitative agreement with results obtained from EDQNM as illustrated in figure 12(b) taken from Lesieur (1990). While the issue of a finite-time enstrophy blow-up is still being debated by the turbulence community (cf. Pumir & Siggia 1990), one thing is clear: the enstrophy grows dramatically when $\nu = 0$. In contrast to the results shown in figure 12(a, b), the commonly used dissipation rate model (68) erroneously predicts that the enstrophy is conserved in the inviscid limit, i.e. that

$$\omega^2 = \text{constant}, \tag{72}$$

when $\nu = 0$. It thus appears that the complete self-preserving solution allows a better treatment of non-equilibrium isotropic turbulence that could be of future use in the development of improved turbulence models.

5. Conclusions

The energy decay for complete self-preserving isotropic turbulence has been re-examined from a basic theoretical and computational standpoint. Several interesting conclusions can be drawn from these results:

(i) The nonlinear differential equations for the energy decay have two fixed points that are stable nodes: $R_{t_\infty} = 0$ and $R_{t_\infty} = \frac{135}{49}(\frac{7}{15}G_0 - 2)^2/S_{K_0}^2$. The former fixed point is only achieved in the limit as $t \to \infty$ and hence is associated with the final period of decay. Consistent with the Batchelor (1948) result, a $K \sim t^{-\frac{5}{2}}$ power law decay is obtained when Loitsianskii's invariant or the Gaussianity of $\tilde{f}(\eta)$ is invoked.

(ii) The non-zero fixed point $R_{t_\infty} = \frac{135}{49}(\frac{7}{15}G_0 - 2)^2/S_{K_0}^2$ is approached within a few eddy turnover times and gives rise to a $K \sim t^{-1}$ asymptotic power law decay. It is the high-Reynolds-number asymptotic solution for a complete self-preserving isotropic turbulence. This solution appears to have been prematurely dismissed by Batchelor (1948) purely on the grounds that Loitsianskii's integral was not an invariant – a constraint which was later found to be violated in isotropic turbulence when $R_{t_0} \gg 1$.

(iii) The structure of the high-Reynolds-number self-preserving solution during the first few eddy turnover times was examined in detail. By a perturbation analysis, it was argued that these solutions can serve as an approximation for the asymptotic approach to a state of complete self-preservation. It was found that, depending on the initial conditions, the early time solutions could be fitted with a power law decay which has an exponent varying from 1.0 to 1.4 – a range of values that is compatible with existing experimental data. Consequently, existing experiments cannot rule out the possibility of a complete self-preserving solution with a $K \sim t^{-1}$ asymptotic power law decay at high Reynolds numbers – a conclusion consistent with the recent experiments of Walker (1986).

(iv) The alternative self-preserving solution of George (1987, 1992) does yield a power law decay with an exponent that depends on the initial conditions; it is a physically consistent solution for the approach to a state of complete self-preservation that can be valid for several eddy turnover times until the skewness peaks.

(v) Since the assumption of complete self-preservation requires that G be constant – and since for high-Reynolds-number isotropic turbulence $\frac{7}{15}G > 2$, whereas for low-Reynolds-number isotropic turbulence $\frac{7}{15}G < 2$ – it is clear that the entire process of isotropic decay from high-Reynolds-number initial conditions to the final period of decay cannot be described by the theory.

Within the framework of self-preservation, the physical origin of a $K \sim t^{-1}$ power law decay becomes clear: it is the asymptotic state toward which a high-Reynolds-number isotropic turbulence is driven in order to resolve an $O(R_t^{\frac{1}{2}})$ imbalance between vortex stretching and viscous diffusion. The resolution of this imbalance also yields compatibility with Kolmogorov scaling wherein $G \sim R_t^{\frac{1}{2}}$. Results were presented which indicate that the complete self-preserving solution yields a better description of non-equilibrium isotropic turbulence than the commonly used turbulence models. It is also interesting to note that when the self-preserving assumption is extended to homogeneous shear flow, a production-equals-dissipation equilibrium can occur – preceded by a transient where K and ϵ grow exponentially – as recently shown by Bernard & Speziale (1992). It thus appears that the theory of self-preservation in homogeneous turbulence has many interesting features that have not yet been fully understood and are worthy of further study.

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