

Atoms trapped in Bessel beams

I. Introduction

Bessel beams appear as the non-diffracting plane wave solutions to the cylindrical wave equation in free space [1]. It has been shown that for a plane wave propagating along the z-axis $E(\rho, \phi, z, t) = f(\rho, \phi) e^{i(k_z z - \omega t)}$, the transverse profiles that satisfy the free space cylindrical wave equation $\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) E = 0$ correspond to $f(\rho, \phi) = J_l(k_\rho \rho) e^{il\phi}$. An exceptional property of such beams is their superluminal phase and group velocities [2]. This can be readily seen by observing that $k_\rho^2 + k_z^2 = \omega^2 / c^2$, thus $\omega / k_z > c$.

Some methods to generate such beams are by placing a circular slit in the focal plane of a lens [1], by letting a Laguerre-Gauss (LG) beam pass through an axicon [3] or by using a Spatial light modulator (SLM).

Such diffraction free beams have been employed as pump fields in non-linear optics experiments. They are also notably useful as waveguides for transporting atoms. Atoms can be trapped in the hollow regions of the beam by blue-detuning the laser frequency or in the intense regions by red-detuning the laser frequency. Here, we look at the atoms trapped in a J1 Bessel potential and compare the two situations of blue and red detuned potentials.

II. Method

To look at the atomic behavior in a Bessel beam we closely follow the approach outlined in [4]. We use the 4th order Runge-Kutta (RK) method to solve the Schrodinger equation for an atom trapped in a Bessel potential given as

$$-\frac{\hbar^2}{2M} \nabla^2 \Psi + V(\rho) \Psi = E \Psi$$

where,

$$\begin{aligned} V(\rho) &= V_0 J_1^2(\rho) \text{ for } \rho < \rho_1 \\ &= V_0 J_1^2(\rho_1) \text{ for } \rho > \rho_1 \end{aligned}$$

for a blue detuned beam. Using separation of variables we express the solution as

$$\Psi(\rho, \phi) = R(\rho) \Phi(\phi),$$

where we have $\Phi(\phi) = e^{im\phi}$ and the radial equation

$$-\frac{\hbar^2}{2M} \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{m^2}{\rho^2} R \right) + V(\rho)R = ER$$

We first scale our units appropriate to the problem, defining $\rho_0 = 1 \mu\text{m}$ and $E_0 = \frac{\hbar^2}{M\rho_0^2}$. For the case of Rubidium (Rb), these values are given as $E_0(^{87}\text{Rb}) = 7.698 \times 10^{-32} \text{ J}$ and $E_0(^{85}\text{Rb}) = 7.879 \times 10^{-32} \text{ J}$ respectively. In the scaled units the radial equation becomes

$$-\frac{1}{2} \left(\frac{d^2 R(\tilde{\rho})}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{dR(\tilde{\rho})}{d\tilde{\rho}} - \frac{m^2}{\tilde{\rho}^2} R(\tilde{\rho}) \right) + \tilde{V}(\tilde{\rho})R(\tilde{\rho}) = \tilde{E}R(\tilde{\rho})$$

with $\tilde{\rho} \equiv \rho / \rho_0$, $\tilde{V}(\tilde{\rho}) \equiv V(\tilde{\rho}) / E_0$ and $\tilde{E} \equiv E / E_0$.

a. Boundary Solutions

Before we apply the Runge-Kutta method we need to have a starting solution for the radial wavefunction and its derivative. We first take the limit $\tilde{\rho}$ going to zero, this gives the modified radial equation for small $\tilde{\rho}$ as

$$-\frac{1}{2} \left(\frac{d^2 R(\tilde{\rho})}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{dR(\tilde{\rho})}{d\tilde{\rho}} - \frac{m^2}{\tilde{\rho}^2} R(\tilde{\rho}) \right) \approx \tilde{E}R(\tilde{\rho})$$

the solution to which are Bessel functions. We therefore have for small $\tilde{\rho}$,

$$R(\tilde{\rho}) \approx J_m \left(\sqrt{2\tilde{E}}\tilde{\rho} \right)$$

$$R'(\tilde{\rho}) \approx J'_m \left(\sqrt{2\tilde{E}}\tilde{\rho} \right)$$

For large $\tilde{\rho}$, we assume the potential to assume a constant asymptotic value \tilde{V}_a

$$R(\tilde{\rho}) \approx Ae^{-\kappa\tilde{\rho}}$$

$$R'(\tilde{\rho}) \approx -\kappa Ae^{-\kappa\tilde{\rho}}$$

such that $\kappa \equiv \sqrt{2(\tilde{V}_a - \tilde{E})}$.

b. Conversion to first order equations

In order to apply the RK method we first convert our second order differential equation into two first order differential equations by defining

$$y_1(\tilde{\rho}) \equiv R(\tilde{\rho})$$

$$y_2(\tilde{\rho}) \equiv R'(\tilde{\rho})$$

This leads to the following equations

$$\frac{dy_1}{d\tilde{\rho}} = f_1(\tilde{\rho}, y_1, y_2)$$

$$\frac{dy_2}{d\tilde{\rho}} = f_2(\tilde{\rho}, y_1, y_2)$$

where the derivatives are defined to be,

$$f_1(\tilde{\rho}, y_1, y_2) \equiv y_2$$

$$f_2(\tilde{\rho}, y_1, y_2) \equiv 2\left(\tilde{V}(\tilde{\rho}) + \frac{m^2}{2\tilde{\rho}^2} - \tilde{E}\right)y_1 - \frac{1}{\tilde{\rho}}y_2$$

c. Applying the RK method

Knowing the starting solutions at the boundaries and the derivatives we can now apply the RK method to obtain the energy eigenvalues and wavefunctions. Discretizing the radial coordinate as $\tilde{\rho}_i = \tilde{\rho}_0 + i\Delta\tilde{\rho}$ we obtain the solution to the defined variables y_1 and y_2 at each grid point as

$$y_1(\tilde{\rho}_{i+1}) \equiv y_1(\tilde{\rho}_i) + \Delta\tilde{\rho}\left(k_1^{(1)} + 2k_1^{(2)} + 2k_1^{(3)} + k_1^{(4)}\right)/6$$

$$y_2(\tilde{\rho}_{i+1}) \equiv y_2(\tilde{\rho}_i) + \Delta\tilde{\rho}\left(k_2^{(1)} + 2k_2^{(2)} + 2k_2^{(3)} + k_2^{(4)}\right)/6$$

with the slopes defined as (n=1,2)

$$k_n^{(1)} = f_n(\tilde{\rho}_i, y_1, y_2)$$

$$k_n^{(2)} = f_n\left(\tilde{\rho}_i + \Delta\tilde{\rho}/2, y_1 + \Delta\tilde{\rho}k_1^{(1)}/2, y_2 + \Delta\tilde{\rho}k_2^{(1)}/2\right)$$

$$k_n^{(3)} = f_n\left(\tilde{\rho}_i + \Delta\tilde{\rho}/2, y_1 + \Delta\tilde{\rho}k_1^{(2)}/2, y_2 + \Delta\tilde{\rho}k_2^{(2)}/2\right)$$

$$k_n^{(4)} = f_n\left(\tilde{\rho}_i + \Delta\tilde{\rho}, y_1 + \Delta\tilde{\rho}k_1^{(3)}, y_2 + \Delta\tilde{\rho}k_2^{(3)}\right)$$

Now, starting at the initial solution for small $\tilde{\rho}$ we obtain the solutions to y_1 and y_2 at large $\tilde{\rho}$ values. If the trial energy eigenvalue we have guessed is correct, then the wavefunction would vanish at large $\tilde{\rho}$. To illustrate the method we look at a test case of the step potential with

$$V(\rho) = 0 \quad \text{for } \rho < \rho_1$$

$$= V_a \quad \text{for } \rho > \rho_1$$

The following plot shows the asymptotic behavior of the radial wavefunction at large $\tilde{\rho}$ for two trial energy lying on either side of the correct energy eigenvalue. Thus whenever we find $y_1(\infty)$ going through zero as we vary our trial energy, we conclude that we would have crossed the energy eigenvalue.

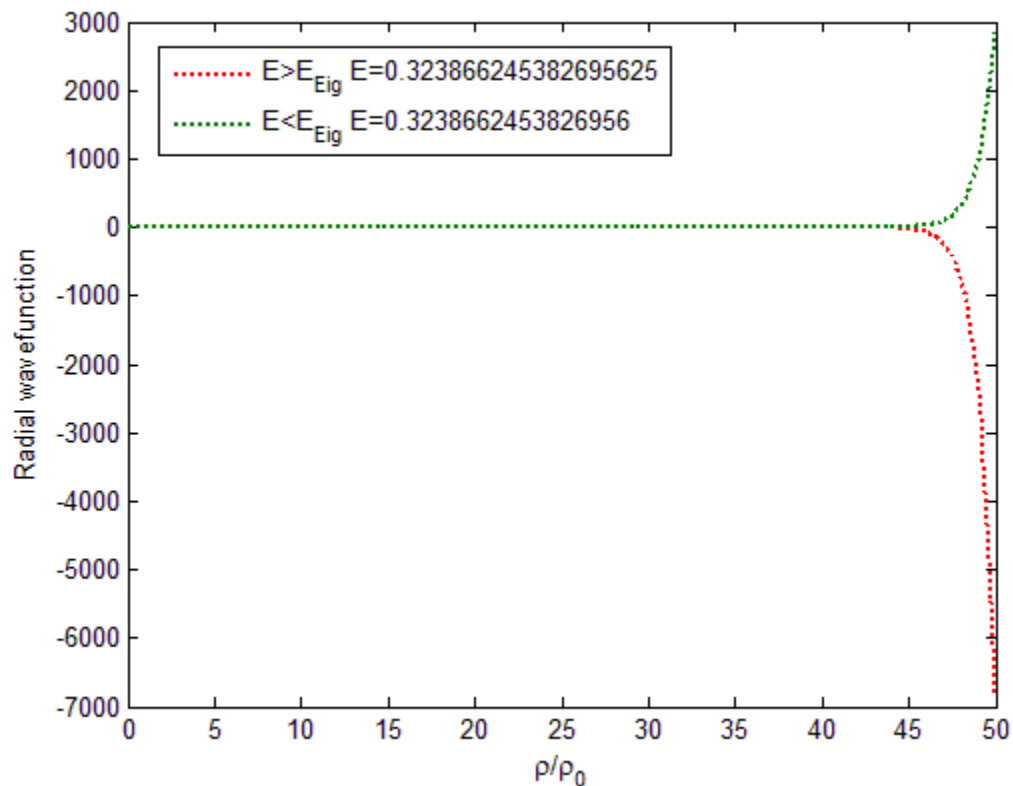


Fig. 1 Radial wavefunction plotted for two trial energies lying on either side of the energy eigenvalue

Using this technique we can locate the energy eigenvalues for any given potential. Now, to obtain the wavefunction, we start at both the ends of the radial coordinate grid with the starting solutions being known and match the wavefunction values at the midpoint. If the slope values match then we have the correct wavefunction, if not then we change try another energy value. It can be seen in the following figure for the step potential case

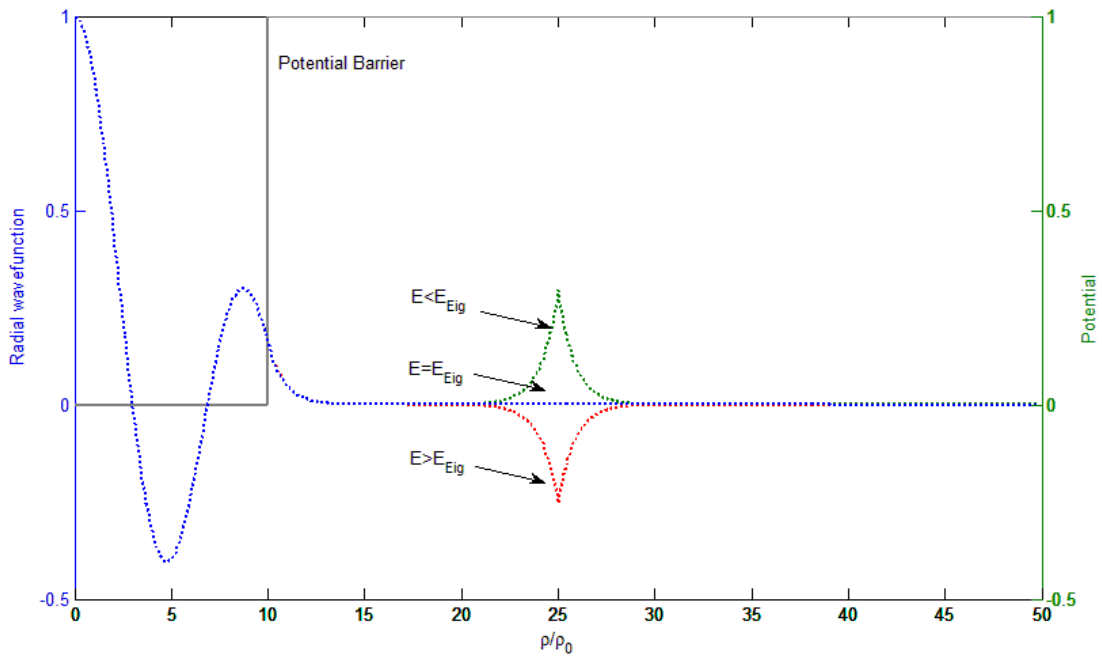
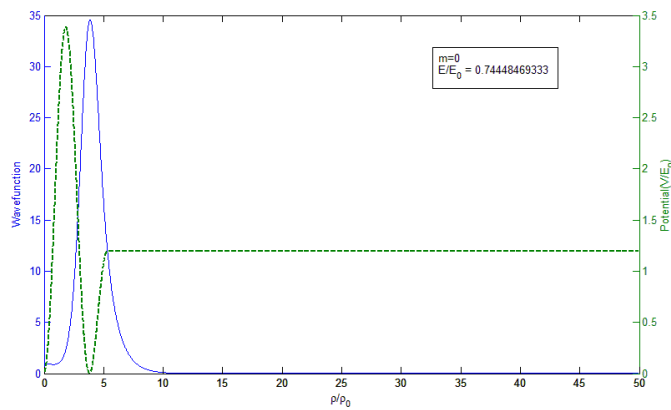


Fig. 2 Matching the slope of radial wavefunction at the midpoint to obtain the eigenfunction

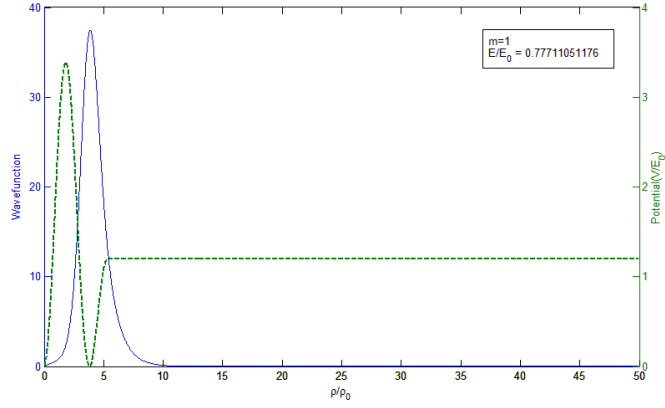
Having outlined the technique of obtaining the solutions for eigenenergies and eigenfunctions, we turn to the particular cases of blue and red detuned Bessel beams.

III. Blue Detuned Bessel beam

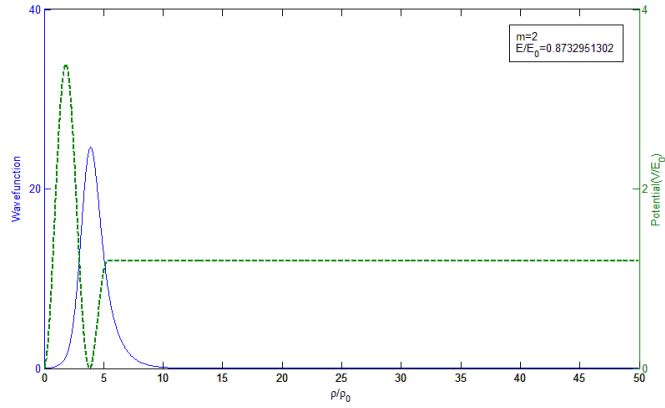
Since we have assumed while obtaining the asymptotic solution that the potential attains a constant value for large $\tilde{\rho}$. We look at the case of an artificial blue detuned beam by including one maxima of the potential function.



(a)



(b)



(c)

Fig. 3 Wavefunctions for Blue detuned Potential (a) $m=0$, $E=0.74484 E_0$ (b) $m=1$, $E=0.77711 E_0$ (c) $m=2$, $E=0.87329 E_0$

We thus find that for a blue detuned potential the wavefunction tends to leak out as the probability density is concentrated near the barrier. On including more number of maxima it is seen that the probability density is larger as one moves radially outward.

IV. Red Detuned Bessel Beam

For the case of red detuned Bessel beam we see that the solution at small r values would change to

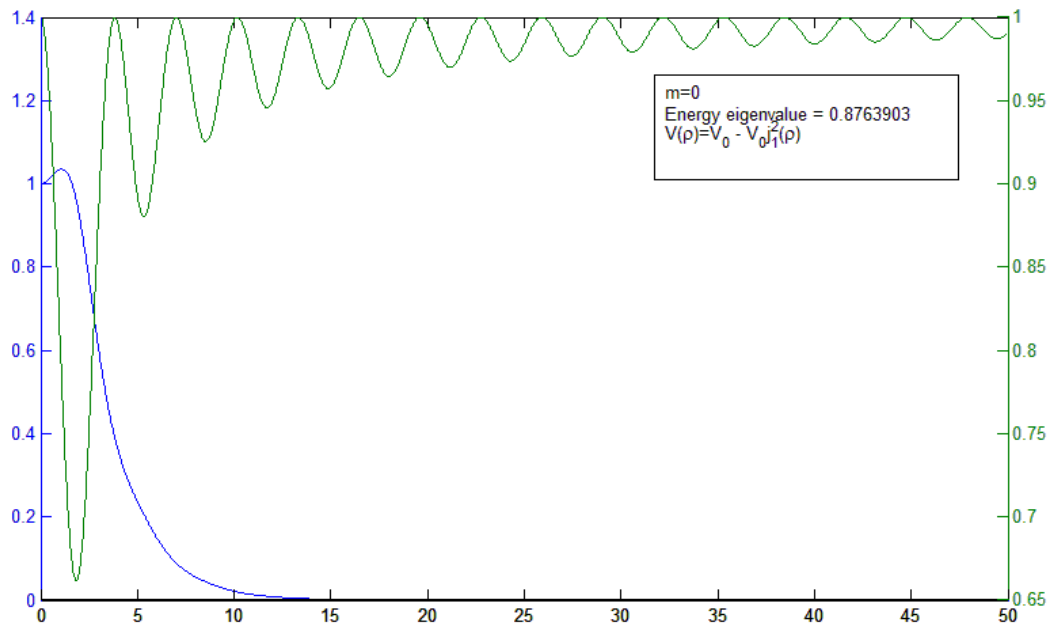
$$R(\tilde{\rho}) \approx I_m\left(\sqrt{2|\tilde{E}|}\tilde{\rho}\right)$$

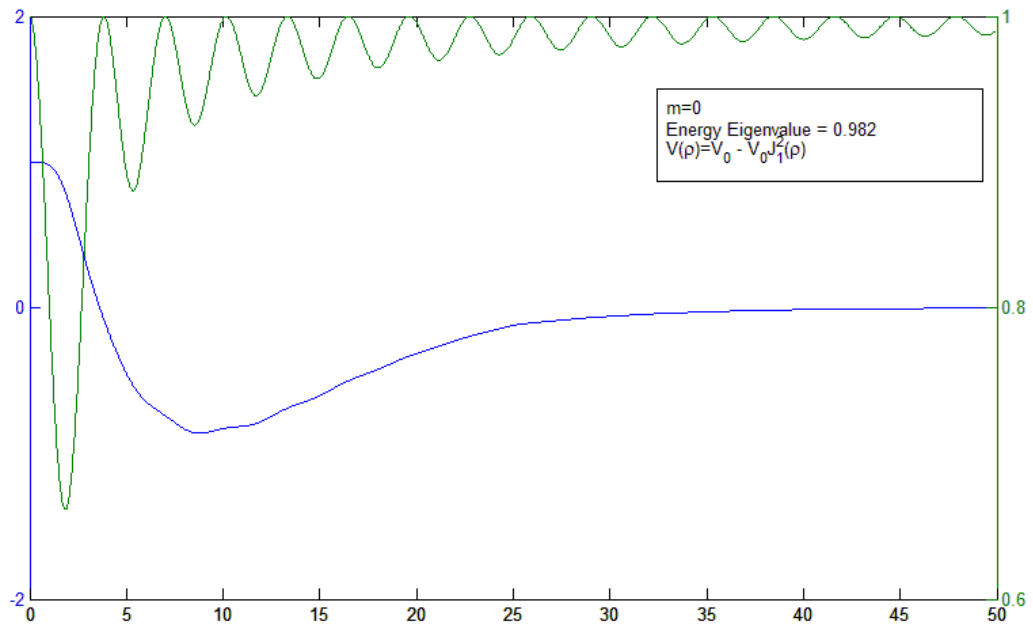
$$R'(\tilde{\rho}) \approx I_m'\left(\sqrt{2|\tilde{E}|}\tilde{\rho}\right)$$

as the energy eigenvalue would now be negative, the potential being

$$V(\rho) = -V_0 J_1^2(\rho)$$

We obtain the following eigenfunctions





V. References

[1]Durnin, J. Opt. Soc. Am. A, **4**, 4, 651 (1987)

[2]Durnin et al, Phys. Rev. Lett., 58, **15**, (1987)

[3]J. Arlt et al, Opt. comm., **177**, 297 (2000)

[4]Notes on Eigenfunctions and Eigenenergies in a Bessel Potential by Mark Edwards and Jeff Heward