Application of Reciprocity to the Design of Electron Guns*

T.M. Antonsen§, Jr., D. Chernin, and J. Petillo
Leidos, Inc. Billerica, MA
§Dept. ECE University of Maryland

* Work supported by DARPA/MTO. The views, opinions and/or findings expressed are those of the authors and should not be interpreted as representing the official views or policies of the Department of Defense or the U.S. Government. Approved for Public Release, Distribution Unlimited
Global Beam Sensitivity Function for Electron Guns  

**Goal**
Derive and Calculate a function that gives the variation of specific beam parameters to

- variations in **electrode potential/**position
- variations in **magnet current/**position

Can be used to
- establish **manufacturing tolerances**
- optimize **gun designs**

Should be **embedded in gun code** (e.g. Michelle)
Basic question: How do small changes in position or potential of anode affect the properties of the beam leaving the gun?

Conventional approach: trial and error. Do many simulations with different anode potentials or positions select the best based on some metric measured at the exit.
Reciprocity - Adjoint Approach

Problem #1

\[ \delta \Phi_A(x) \] Due to wall displacement

\[ \text{Change in beam radius} \]

Problem #2

\[ \delta E_n \] Calculate and record change in normal E.

\[ \text{Reverse and perturb electron coordinates} \]

Electrons run backwards

\[ \delta E_n \] Is the sensitivity function
EM Reciprocity

Example:
- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity
Basic Formulation – Linear Algebra

Nonlinear System: \[ A_{NL}(x) = B \]
Metric \[ M(x|B) \]

\( x = \text{state} \) \( B = \text{parameters} \) \( M = \text{metric} \)

Small changes of the parameters:
\[ A \cdot \delta x = \delta B \]
for many \( B \)’s.

And then evaluate for each \( B \):
\[ \delta M = C \cdot \delta x^\dagger \]

\( \delta M \) is the answer.

Instead solve for \( y \) once:
\[ A^\dagger \cdot y = C \]

Then:
\[ \delta M = \delta B^\dagger \cdot y \]
Optimize shape to minimize drag.

Result is also aesthetically appealing.

Super Computer
1985 Volvo 240 DL
2017 Porche Panamera

That’s more like it !!!
Code (Michelle) solves the following equations:

Equations of motion for N particles $j=1,N$

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x}$$

Accumulates a charge density

$$\rho(x) = \sum_j^{T_j} I_j \int_0^{T_j} dt \delta(x - x_j(t))$$

Solves Poisson E

$$-\nabla^2 \Phi = 4\pi\rho$$

Iterates until converged

Michelle: Petillo, J; Eppley, K; Panagos, D; et al., IEEE TPS 30, 1238-1264 (2002).


Reciprocity - Adjoint Approach

Problem #1

$$\delta \Phi_A(x)$$  Due to wall displacement

Change in beam radius

Problem #2

$$\delta E_n$$  Calculate and record change in normal E.

Reverse and perturb electron coordinates

Electrons run backwards

$$\delta E_n$$  Is the sensitivity function
Reference Solution + Two Linearized Solutions

\[
\begin{align*}
(x_j, p_j) &\rightarrow (x_j, p_j) + (\delta x_j, \delta p_j) \\
\rho(x) &\rightarrow \rho(x) + \delta \rho(x) \\
\Phi(x) &\rightarrow \Phi(x) + \delta \Phi(x)
\end{align*}
\]

Reference Solution \hspace{1cm} Perturbation

Two Linearized Solutions

\[
[\delta x_j(t), \delta p_j(t)] \hspace{0.5cm} \text{true}
\]

\[
[\delta \hat{x}_j(t), \delta \hat{p}_j(t)] \hspace{0.5cm} \text{adjoint}
\]

subject to different BC’s

Can show

\[
\sum_j I_j \left( \delta \hat{p}_j \cdot \delta x_j - \delta p_j \cdot \delta \hat{x}_j \right) \bigg|_0^{T_j} = -\frac{q}{4\pi} \int_S d\mathbf{n} \cdot \left[ \delta \Phi \nabla \delta \hat{\Phi} - \delta \hat{\Phi} \nabla \delta \Phi \right]
\]
Hamilton’s Equations $H(p,q,t)$

Conserve Symplectic Area

\[
\begin{align*}
\frac{d\delta q_1}{dt} &= \frac{\partial^2 H}{\partial p \partial q} \cdot \delta q_1 + \frac{\partial^2 H}{\partial p \partial p} \cdot \delta p_1 \\
\frac{d\delta p_1}{dt} &= -\frac{\partial^2 H}{\partial q \partial q} \cdot \delta q_1 - \frac{\partial^2 H}{\partial q \partial p} \cdot \delta p_1 \\
\frac{d\delta q_2}{dt} &= \ldots \\
\frac{d\delta p_2}{dt} &= -\ldots
\end{align*}
\]

\[
\frac{d}{dt}(\delta p_1 \cdot \delta q_2 - \delta p_2 \cdot \delta q_1) = 0
\]

Area conserved for any choice of 1 and 2
Main Result: Generalization of Green’s Theorem

\[
\sum_j I_j \left( \delta \hat{p}_j \cdot \delta x_j - \delta \hat{x}_j \cdot \delta p_j \right) \bigg|_0^{T_j} = -\frac{q}{4\pi} \int_S d\mathbf{n} \cdot \left[ \delta \hat{\Phi} \nabla \delta \hat{\Phi} - \delta \hat{\Phi} \nabla \delta \Phi \right]
\]
Can show

\[ \sum_{j} I_{j} \left( \delta \hat{p}_{j} \cdot \delta x_{j} - \delta p_{j} \cdot \delta \hat{x}_{j} \right) \bigg|_{0}^{T_{j}} = -\frac{q}{4\pi} \int_{S} d\mathbf{n} \cdot \left[ \delta \Phi \nabla \delta \hat{\Phi} - \delta \hat{\Phi} \nabla \delta \Phi \right] \]

**Problem #1** (true problem) Unperturbed trajectories at cathode, Perturbed potential on boundary.

\[ \delta p_{j} \bigg|_{0} = 0, \quad \delta q_{j} \bigg|_{0} = 0, \quad \delta \Phi(x) \neq 0 \]

\[ \sum_{j} I_{j} \left( \delta \hat{p}_{j} \cdot \delta x_{j} - \delta p_{j} \cdot \delta \hat{x}_{j} \right) \bigg|_{\text{Exit}} = -\frac{q}{4\pi} \int_{S} d\mathbf{n} \cdot \left[ \delta \Phi \nabla \delta \hat{\Phi} - \delta \hat{\Phi} \nabla \delta \Phi \right] \]

**Problem #2** (adjoint problem) Perturbed trajectories at exit, Unperturbed potential on boundary.

\[ \delta \hat{p}_{j} \bigg|_{T} = \lambda \mathbf{x}_{\perp j}, \quad \delta q_{j} \bigg|_{T} = 0, \quad \delta \hat{\Phi}(x) = 0 \]

\[ \lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_{j} I_{j} \left( \mathbf{x}_{j} \cdot \delta \mathbf{x}_{j} \right) \bigg|_{T_{j}} = -\frac{q}{4\pi} \int_{S} d\mathbf{a} \delta \Phi \left( \mathbf{n} \cdot \nabla \delta \hat{\Phi} \right) \]

Sensitivity Function
Sensitivity for RMS Beam Radius

\[
\lambda \sum_j I_j (x_j \cdot \delta x_j)_{T_j} = -\frac{q}{4\pi} \int da \delta \Phi (n \cdot \nabla \delta \hat{\Phi})
\]

\[
= \lambda I R_{RMS} \delta R_{RMS}
\]

Sensitivity Function
Numerical Accuracy – Changing the Anode Voltage

Blue: True Problem,
RMS Radius vs Anode Potential

Red: Adjoint Problem,
RMS Radius vs Coordinate Perturbation
Conclusion: Next Steps

Add Magnetic field

\[
\sum I_j \left( \delta \hat{p}_j \cdot \delta \mathbf{x}_j - \delta p_j \cdot \delta \hat{\mathbf{x}}_j \right)_{T_j} = -q \varepsilon_0 \int_S d a \delta \Phi_A \mathbf{n} \cdot \nabla \delta \hat{\Phi}_S - \mu_0 \int d^3 x \delta j_m \cdot q \delta \hat{A}_s
\]

Other Metrics: e.g. emittance

Add time dependence (done for time periodic)

Implement in an optimization routine
Next Steps

- Consider other goal functions
  - RMS radius
  - Emittance
  - Deviations from Brillouin Flow

- Add magnetic Field
- 3D
- Sensitivity to Magnet Strength/Placement

\[
\sum_{j} l_{j} \left( \delta \hat{p}_{j} \cdot \delta x_{j} - \delta p_{j} \cdot \delta \hat{x}_{j} \right) = -\frac{q}{4\pi} \int_{S} d\alpha \delta \Phi_{A} n \cdot \nabla \delta \hat{\Phi} - \int d^{3}x \delta j_{m} \cdot q \delta \hat{A}_{s}
\]

sensitivity function

Change in magnetization current

Thank You
Example of the Adjoint Method in Action

Problem: Compute the displacement of the beam in a sheet beam gun due to a small change in anode potential or a small displacement of the anode:

\[ \int \Phi \nabla \Phi \cdot \delta n = s \]

\[ \delta x = \text{Beam centroid displacement at gun exit} \]

\[ \delta x = - \frac{q}{4\pi \lambda I} \int \delta n \cdot \nabla \phi \nabla \phi \]

\(-n \nabla \phi = \text{Sensitivity (Green's) function}\)

MICHELLE Simulations of Sheet Beam Gun

‘Perfect’ case:
Beam centroid at gun exit is on axis

‘Perturbed’ case:
Beam centroid at gun exit is displaced

The adjoint method gives us a way to compute the displacement of the beam \textit{without} re-running MICHELLE:
Theoretical Study of Statistical Variations
Successful Test of Adjoint Method!

Vector plot of the ‘sensitivity’ or Green’s function

\[ -\nabla \delta \Phi \]

‘Direct’ MICHELLE Simulation with Perturbed Anode Voltages

\[ \delta x_{\text{actual}} \]

\[ \delta x_{\text{pred}} = -\frac{q}{4\pi\lambda I} \int \! dA \mathbf{n} \cdot \delta \Phi \nabla \delta \hat{\Phi} \]

Comparison: predicted displacement/actual displacement

\[ \frac{\delta x_{\text{pred}}}{\delta x_{\text{actual}}} = 0.9969 \]
Effective Area – Antenna Gain

\[ P_R = A_e(\Omega)I \]

Effective area

\[ A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi} \]

Incident intensity

\[ G(\Omega) = \frac{dP}{d\Omega} / P_T \]

Power per unit solid angle

\[ P_T = \int \frac{dP}{d\Omega} d\Omega \]