Introduce concept for plasma waves

\[ U = \frac{E^2}{8\pi} + \frac{1}{2} m v^2 \text{c} \]

**Wave Energy & Momentum**

**Maxwell's Equations**

\[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \cdot E = 4\pi \rho \]

\[ \nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t} \]

\[ E \cdot \nabla \times B - B \cdot \nabla \times E = -\nabla \cdot E \times B \]

\[ = \frac{4\pi}{c} E \cdot J + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{E^2 + B^2}{2} \right) \]

\[ \frac{\partial}{\partial t} \frac{E^2 + B^2}{8\pi} + \nabla \cdot \left( \frac{C}{4\pi} E \times B \right) = -E \cdot J \]

**Poynting's Theorem**

\[ \frac{\partial}{\partial t} U + \nabla \cdot S = -E \cdot J \]

**Electromagnetic Power Flux**

Work done by fields on currents
Similarly for momentum

\[ \frac{d}{dt} (P_{\text{mech}} + P_{\text{field}}) = +\nabla \cdot T \]

Where

\[ P_{\text{field}} = \frac{1}{4\pi c} E \times B \quad P_{\text{mech}} = \frac{1}{6} m v^2 \]

\[ T = \frac{1}{4\pi} \left[ E E + B B - \frac{1}{2} \frac{1}{\omega} (E^2 + B^2) \right] \]

Why is this description less ideal for waves in dispersive media?

Ans: Because it keeps separate accounts of particle energy and electromagnetic field energy.

The energy density (and momentum density) of a wave \[ \bar{\varepsilon} \] has two components: the
ELECTROMAGNETIC ENERGY DENSITY \( \frac{E^2 + B^2}{8\pi} \)

AND THE ENERGY DENSITY OF THE COHERENT MOTION OF THE PLASMA PARTICLES. WHY IS IT IMPORTANT TO COMBINE THESE TWO? 

ANS: SUPPOSE I WANT TO MAKE A WAVE WITH GIVEN ELECTRIC FIELD AMPLITUDE \( E \), THE AMOUNT OF ENERGY THAT I MUST INVEST IS THE SUM OF THE ELECTROMAGNETIC AND MECHANICAL ENERGIES. SIMILARLY, IF A WAVE WITH GIVEN AMPLITUDE DUMPS THE AMOUNT OF ENERGY THAT MUST BE DISSIPATED IS AGAIN THE SUM OF THE TWO COMPONENTS. (SIMILAR ARGUMENTS APPLY FOR MOMENTUM)

FOR THESE REASONS WE WOULD LIKE AN EXPRESSION FOR THE ENERGY DENSITY, MOMENTUM DENSITY AND POWER FLUX OF A WAVE EXPRESSED IN TERMS OF THE AMPLITUDE OF THE WAVE ELECTRIC FIELD.

IN PRINCIPLE THESE CAN BE OBTAINED FROM THE DISTRIBUTION FUNCTION HOWEVER THERE IS AN EASIER WAY.
Wave Propagation in an Infinite Homogeneous Plasma

Electric Field

\[ E = \mathcal{R}(\hat{E} \exp(ik \cdot x - i\omega t)) + \text{c.c.} \]

\[ B = \hat{B} \]

Maxwell's Equations

\[ \mathbf{k} \times \hat{E} = \frac{c}{\varepsilon} \hat{B} \]

\[ \mathbf{i}k \cdot \hat{E} = 4\pi \rho \]

\[ \mathbf{i}k \times \hat{B} = \frac{4\pi}{c} \hat{J} - \frac{i\omega}{c} \hat{E} \]

\[ \mathbf{i}k \cdot \hat{B} = 0 \]

Dielectric Tensor

\[ \frac{\mu}{c^2} \hat{J} - \frac{i\omega}{c} \hat{E} = -\frac{i\omega}{c} \hat{K} \cdot \hat{E} \]

In General

\[ \hat{K} = \hat{K}(k, \omega) \]
\[
k \times \frac{e}{\omega} k \times \hat{E} = -\frac{\epsilon}{\omega} k \times \hat{E}
\]

\[
\frac{c^2}{\omega^2} k \times (k \times \hat{E}) + \epsilon k \cdot \hat{E} = 0
\]

\[
G \cdot \hat{E} = 0 \quad \text{DISPERSION RELATION} \quad \det |G| = 0
\]

\[
\text{FOR COLD PLASMA}
\]

\[
\tilde{\epsilon} = \frac{1}{\tau_{\text{ee}}} \left[ \begin{array}{ccc}
\frac{\omega_p^2}{\omega^2 - \Omega_0^2} & \frac{i \omega_p^2 \Omega_0}{\omega (\omega^2 - \Omega_0^2)} & 0 \\
-\frac{i \omega_p^2 \Omega_0}{\omega (\omega^2 - \Omega_0^2)} & \frac{\omega_p^2}{\omega^2 - \Omega_0^2} & 0 \\
0 & 0 & \frac{\omega_p^2}{\omega^2} \end{array} \right]
\]

\[
\omega_p^2 = \frac{4 \pi n_0 e_0^2}{m_0} \quad \Omega_0 = \frac{e_0 B}{m_0 c}
\]

\[
\text{NOTE: FOR COLD PLASMA } k_x \text{ IS INDEPENDENT OF } k_x \text{ HERMITIAN } k^+ = k^* \]
METHOD OF DETERMINING ENERGY DENSITY OF WAVE

Instead of computing separately the electromagnetic / mechanical portions of the wave energy density lets ask "How much work do I have to do to create a wave of given amplitude". To do this lets introduce a test current and associated charge density which is separate from the plasma currents and charges.

That is

\[ \nabla \cdot \mathbf{E} = 4\pi \rho + 4\pi \rho_T \]

Plasma Test Charge

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c^2} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}_T \]

Plasma Test Current
AND LET

\[ x_T = \Re \left\{ \int_{x_T}^{\infty} \exp(ik \cdot x - i(w + i\varepsilon)t) \right\} \]

- \(i(w + i\varepsilon) \hat{\rho}_T + ik \cdot \hat{J}_T = 0\)

CONTINUITY OF TEST CHARGE

WHAT IS THE WORK THAT THE TEST CURRENT MUST DO ON THE PLASMA to build up a wave of a given amplitude?
The work done by force on \( J_T \)

\[
W = -\int_{-\infty}^{t} \, dt' \, j_T \cdot E
\]

\[
= -\frac{1}{4\pi} \int_{-\infty}^{0} \, dt' \, \left[ \hat{j}_T \cdot \hat{E}^* + \hat{j}_T^* \cdot \hat{E} \right] \exp(2\pi i t')
\]

\[
= -\frac{1}{8\pi} \left[ \hat{j}_T \cdot \hat{E}^* + \hat{j}_T^* \cdot \hat{E} \right]
\]

**From Maxwell's Equations**

\[
i k \times \frac{1}{\omega^2} \hat{\nabla} \times \hat{E} + \frac{i \omega}{c} \hat{k} \cdot \hat{E} = \frac{4\pi}{c} \hat{j}_T
\]

\[
= \frac{i \omega}{c} \hat{G} \cdot \hat{E} = \frac{4\pi}{c} \hat{j}_T
\]

\[
G = \frac{\mathcal{E}}{\varepsilon} - \frac{i}{\omega} \frac{k \cdot \varepsilon}{\omega^2} + \frac{k \cdot k \varepsilon}{\omega^3}
\]

\[
W = -\frac{1}{8\pi} \left[ \frac{i \omega}{4\pi} \hat{G} \cdot \hat{E}^* \hat{E} - \frac{i \omega^*}{4\pi} \hat{E} \cdot \hat{G}^* \hat{E}^* \right]
\]
\[ W = -\frac{1}{8\pi} \frac{E^*}{4\pi} \left[ i(\omega \text{ } \hat{G}) - i(\omega^* \text{ } \hat{G}^*) \right] \cdot E \]

**TAYLOR EXPAND**
\[ \omega \hat{G} \approx \omega_r \hat{G}(\omega_r) + i\gamma \frac{2}{\omega_r} \omega_r \hat{G}(\omega_r) \]

**FOR LOSS FREE MEDIUM:**
\[ \left( \hat{G}^*(\omega_r) \right)^\dagger = \hat{G}(\omega_r) \]

\[ W = \frac{E^*}{16\pi} \frac{\partial}{\partial \omega_r} \omega_r \hat{G}(\omega_r) \cdot E \]

\[ W = \text{ENERGY DENSITY OF WAVE} \]

**FOR DISSIPATING MEDIUM**
\[ (\hat{G}^*(\omega_r))^\dagger \neq \hat{G}(\omega_r) \]
But if \( G^{*+} - G \ll G^{*+} + G \)

(weakly dissipative the expression for \( W \) is still valid)

Components of \( W \)

\[
W = \frac{1}{16\pi} \, \hat{E}^* \cdot \frac{2}{\omega_r} \, \frac{d}{d\omega_r} \left[ \frac{c^2}{\omega_r^2} \, \hat{k} \times \hat{k} \times \hat{E} + \hat{k} \cdot \hat{E} \right]
\]

\[\xi = \frac{1}{\omega_r} + \sqrt{\frac{4\pi n e^2}{m}} \]

PLASMA CONTRIBUTION

L DISPLACEMENT CURRENT

\[
W = \frac{1}{4\pi} \left[ - \hat{E}^* \cdot \hat{k} \times \hat{k} \times \hat{E} + \hat{E}^* \cdot \hat{E} + \hat{E}^* \cdot \frac{\partial}{\partial \omega_r} \frac{\omega_r \hat{E}}{\hat{E}} \right]
\]

\[
\frac{c}{\omega_r} \hat{k} \times \hat{E} = \hat{B}, \quad -\frac{c}{\omega_r} \hat{E}^* \cdot \hat{k} \times \hat{B} = \hat{B} \cdot \frac{c}{\omega_r} \hat{k} \times \hat{E}^* = \hat{B} \cdot \hat{B}^*
\]
\[
W = \frac{1}{4\pi} \left[ \frac{\hat{E} \cdot \hat{E} + \hat{B} \cdot \hat{B}}{\pi} + \frac{\hat{E} \cdot \frac{\partial}{\partial \omega_r} \omega \frac{\hat{E}}{\omega}}{(\epsilon - \frac{1}{2})} \right]
\]

**FIELD CONTRIBUTION**

**PLASMA CONTRIBUTION**

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**MOMENTUM DENSITY OF WAVE**

In a similar manner we may calculate the momentum density of the wave.

What is the rate at which momentum is transferred from the test current to the wave?

\[
\frac{d}{dt} P_t = e_n e_t E + e_t n_t v_t \times B
\]

\[
P_t = \int_{-\infty}^{t} \left[ P_t E + \frac{1}{c^2} j_t \times B \right] dt
\]
\[ P_t = \int_{-\infty}^{t} \left[ \rho_t \hat{E}^* + \rho_t^* \hat{E} + \frac{1}{c} \hat{J}_t \times \hat{B} + \frac{1}{c} \hat{J}_t \times \hat{B}^* \right] e^{\gamma t} \]

\[ i \hbar \cdot \hat{E} = \rho_t \]
\[ \hat{B} = \frac{c}{\omega} \hat{k} \times \hat{E} \]

\[ \frac{1}{c} \hat{J}_t \times \hat{B}^* = \hat{J}_t \frac{1}{\omega^*} \times \hat{k} \times \hat{E}^* \]

\[ = \frac{k}{\omega^*} \hat{J}_t \cdot \hat{E}^* - \hat{E}^* \frac{k \cdot \hat{J}_t}{\omega^*} \]

\[ \omega_r \gg \gamma \]
\[ = \frac{k}{\omega} \hat{J}_t \cdot \hat{E}^* - \rho_t \hat{E}^* \]

\[ P_t = \int_{-\infty}^{t} dt' \exp(2\pi i t') \frac{k}{\omega} \left[ \hat{J}_t \cdot \hat{E}^* + \hat{J}_t^* \cdot \hat{E} \right] \]

\[ P_w = -P_t = \frac{k}{\omega} \]

\[ P_{-w} = -P_t = \frac{k}{\omega} \]
\[ P_w = \frac{k}{\omega} W \]

**Quantum Mechanical Analogy**

\[
\begin{align*}
P_m &= \hbar k_m \\
\varepsilon &= \hbar \omega \\
P &= \frac{k}{\omega} \varepsilon
\end{align*}
\]

Thus, even for a dispersive anisotropic medium, the momentum density and energy density are simply related.
Introducing wave action

$$N_k = \text{the wave 'action'} = \frac{1}{4} \varepsilon_0 E_k^2 \frac{2\varepsilon}{\partial \omega} |\omega = \omega_k$$

$$V_k = \omega_k N_k$$

$$P_k = k N_k$$

analogy with quantum mechanics

the photon energy

the photon momentum

$$N_k \rightarrow$$ number of plasma

The amount of energy to set up the wave.

There are two states of the plasma before the wave that set up.

The energy in plasma $\omega$ should consist from two part: energy of electrostatic field and kinetic energy of particles moving due to this field.

<table>
<thead>
<tr>
<th>Electron</th>
<th>Plasma</th>
<th>Oscillation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}$</td>
<td>$\omega_k = \omega_p$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial^2 \varepsilon}{\partial \omega^2} = \frac{2\varepsilon_p^2}{\omega^3}$</td>
<td>$\frac{\partial \varepsilon}{\partial \omega}</td>
<td>\omega = \omega_k = 2$</td>
</tr>
</tbody>
</table>

$$V_k = \frac{1}{4} \varepsilon_0 |E_k|^2 \cdot 2$$

Time average of electrostatic energy density (one $\frac{1}{2}$ - average over time, another $\frac{1}{2}$ - average of $E$)

Factor 2 shows, that the kinetic energy of the oscillating particles (electrons) = $\frac{1}{4} \varepsilon_0 |E_k|^2$
**Group Velocity vs Power Flux**

**Group Velocity**

Suppose we solve

$$\det (G(w,r)) = 0$$

and obtain the solution

$$w = w(r)$$

The group velocity is defined as

$$\frac{\partial w}{\partial r} = V_g$$

This is the **group velocity** of a wave packet and also the rate at which the energy density is transported.

$$\Gamma = V_g W$$

$$\Gamma = \text{power flux} \left( \frac{\text{ergs}}{\text{sec} \cdot \text{cm}^2} \right)$$
In order to determine $\mu_0$ and $\nu_0$

\[ G \cdot Е = 0 \]

Let $\omega = \omega_0 + \delta \omega$  

\[ k = k_0 + \delta k \]

\[ Е = Е_0 + \delta Е \]

Linearize:

\[ G(\omega_0, k_0) \cdot Е_0 + G(\omega_0, k_0) \cdot \delta Е \]

\[ + \delta \omega \frac{∂}{∂\omega_0} G(\omega_0, k_0) \cdot Е_0 + \delta k \cdot \frac{∂}{∂k_0} G(\omega_0, k_0) \cdot Е_0 = 0 \]

Dot on left with $Е_0^*$ and assume $G$ is Hermitian

\[ Е_0^* \cdot Е_0 = 0 \]

\[ \delta \omega \cdot Е_0^* \cdot \frac{∂}{∂\omega_0} Е_0^* - Е_0 = -\delta k \cdot \frac{∂}{∂k_0} \omega_0 \cdot Е_0^* \cdot Е_0 \]
\[ \delta \omega W = + \delta k \cdot \left\{ - \frac{\omega_0^2}{4\pi} \frac{2}{d^2 k_0} \mathbf{E}_0 \times \mathbf{E}_0 \right\} \]

In other words

\[ \delta \omega W = \delta k \cdot \mathbf{V}_W \]

\[ \mathbf{P} = - \frac{\omega}{4\pi} \frac{\mathbf{\mathbf{E}}}{d^2 k_0} \mathbf{E}_0 \times \mathbf{E}_0 \]

\[ \mathbf{P} = \mathbf{S} + \mathbf{T} \]

\[ \mathbf{S} = \frac{c}{4\pi} \left[ \mathbf{E}_0 \times \mathbf{B}_0 + \mathbf{E}_0 \times \mathbf{B}^* \right] / 4 \text{ Pointing} \]

\[ \mathbf{T} = - \frac{\omega_0^2}{4\pi} \frac{2}{d^2 k_0} \mathbf{E}_0 \times \mathbf{E}_0 / 4 \text{ Particle cont.} \]

\( \text{For cold plasma,} \ \frac{\mathbf{S}}{\mathbf{S}} \text{ is independent of} \ \mathbf{k}, \ \text{all energy flux is} \ \text{Pointing.} \)
Damping: Suppose

\[ G = G^n + iG^a \]

where \((G^h)^* = G^+\)

\[ G^h \gg G^a \]

\[ E = E_0 + \delta E \quad \omega = \omega_0 + \delta \omega \]

\[ \delta \omega = \frac{2}{\omega_0} E_0 \cdot G^h \cdot E^0 + E_0^* \cdot \omega G^a \cdot E^0 = 0 \]

Rate at which energy is dissipated

\[ \text{Damping Rate} = \frac{\omega_0}{4\pi} \frac{E_0^* \cdot G^a \cdot E^0}{W} \]