Class Notes: Lecture I

Review of Landau Damping (Initial Value Solution of the Vlasov Equation)

Assume $B = 0$

\[ \frac{\partial f}{\partial t} + \nabla \cdot \mathbf{v} f + \frac{e}{m} \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad \text{Vlasov Eq.} \]

\[ \nabla \cdot \mathbf{E} = 4\pi q \int d^3 \mathbf{v} f - 4\pi q n_0 \quad \text{Poisson Eq.} \]

Linearize above

\[ E = -\nabla \phi_i \quad \phi_i = \phi_i(t) \exp(i k \cdot x) \]

\[ f = f_0 + f_i \quad f_0 = f_0(\mathbf{v}) \]

\[ f_i = f_i(t) \exp(i k \cdot x) \]
\[
\frac{df_i(t)}{dt} + ik \cdot v \cdot f_i = \frac{q}{m} \phi(t); \quad k \cdot \frac{\partial}{\partial x} f_0
\]

\[
k^2 \phi_i(t) = 4\pi q \int d^3v f_i
\]

**Introduce Laplace Transforms**

\[
g(\omega) = \int_0^\infty dt g(t) \exp(i\omega t) \quad \text{(a)}
\]

\[
g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{g}(\omega) \exp(-i\omega t) \quad \text{(b)}
\]

**Integral (a) is defined only for**

**Particular values of \( \omega \) such that as \( t \to \infty \)

\[
g(t) \exp(i\omega t) \to 0 \quad \text{sufficiently fast. For}
\]
Example, if \( g(t) = g_0 \exp(i\omega_0 t) \), we must have \( \text{Im}(w) > \text{Im}(\omega_0) \) in order that the integral be well defined. In general this will be the case if \( \text{Im}(w) \) is sufficiently large. Integral (b) is carried out along the contour \( L \) which passes from \(-\infty\) to \(+\infty\) through regions of the \( w \)-plane where \( \bar{g}(w) \) is defined. For the case of \( g(t) = g_0 \exp(i\omega_0 t) \) the contour \( L \) must pass above the point \( w = -\omega_0 \).
LAPLACE TRANSFORM EQUATIONS

\[ \int_0^\infty dt \exp(i\omega t) \frac{d}{dt} f_i(t) \]  
\[ = -f_i(t=0) - i \omega \int_0^\infty dt \exp(i\omega t) f_i(t) \]

\[-i(\omega - k \cdot v) \overline{\phi_i}(\omega) = \frac{q}{m} \overline{\phi_i}(\omega) \cdot \frac{k \cdot \nabla \phi_0}{\omega - k \cdot v} + \overline{f_i(t=0)} \]

\[ k^2 \overline{\phi_i}(\omega) = 4\pi q \int d^3v \overline{f_i}(\omega) \]

Solving for \( \overline{f_i}(\omega) \) and substituting in Poisson's Eq. gives

\[ k^2 E(k,\omega) \overline{\phi_i}(\omega) = 4\pi q \int d^3v \frac{f_i(t=0)}{\omega - k \cdot v} \]

WHERE:

\[ E(k,\omega) = 1 + \frac{4\pi q^2}{mk^2} \int d^3v \frac{k \cdot \nabla \phi_0}{\omega - k \cdot v} \]
Inverting $\phi(w)$

$$\phi_i(t) = \int_L \frac{dw}{2\pi} \, 4\pi i \int d^3y \, f(0) \exp(-iw t) \frac{\delta(w - \frac{\vec{k} \cdot \vec{Y})}{k^2 \epsilon(k, w)}$$

To evaluate integral, deform contour and analytically continue integrand into lower half $w$-plane.
On lower portions of contour \( \exp(-i\omega t) \to 0 \) \((t>0)\). Therefore \( \phi(x,t) = -2\pi i \) times residues at singularities.

\[ \varepsilon(\mathbf{k}, \omega_0) = 0 \]

\[ \phi(x,t) = -i \left[ \exp(-i\omega_0 t) \cdot 4\pi q_i \cdot \int d^2 \mathbf{\nu} \frac{f(0)}{(\omega_0 - \mathbf{k} \cdot \mathbf{\nu})} \frac{d\varepsilon}{d\omega_0} \right] \]

\[ + 4\pi q_i \cdot \int d^2 \mathbf{\nu} \frac{f(0) \exp(-i\mathbf{k} \cdot \mathbf{\nu} t)}{\varepsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{\nu})} \]

The first term oscillates at frequency \( \omega_0 \) and represents the excitation of a natural mode of the system by the initial conditions. The second term has a complicated time dependence.
AND REPRESENTS THE WAY IN WHICH THE INITIAL DISURANCE EVOLVES DUE TO THE FREE STREAMING OF THE PARTICLES.

AS T GETS LARGE, THE RAPID OSCILLATIONS IN V OF THE FACTOR $\exp(-ik\cdot x t)$ CAUSE THIS CONTRIBUTION TO BECOME SMALL.

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**Wave crests for initial perturbation**

**Phase space**

**Wave crests for t>0**

**Phase space**

Averaging over V GIVES SMALL CONTRIBUTION
Natural mode of the system occurs for \( \varepsilon(k, \omega_0) = 0 \)

\[
\varepsilon(k, \omega_0) = 1 + \frac{4\pi q^2}{mk^2} \int \frac{d^3v}{\omega - k \cdot v} \frac{k \cdot \frac{\partial f_0}{\partial v}}{\omega_0 - k \cdot v}
\]
do \( v \) interval

Note that in the integral over velocity there is a singularity at \( \omega_0 = \frac{k \cdot v}{2} = kv \)

(\( v \) is the component of \( v \) parallel to \( k \))

Assume \( k > 0 \)

\[ \text{Singularity at } \omega_0 = \frac{k \cdot v}{2} = kv \]

\[ \text{Re}(V_n) \]

\[ \text{Im}(V_n) \]
If $\text{Im}(w_0) > 0$ then the singularity lies above the integration contour (in $v_i$) and the integral is well defined.

If we are required to determine $\phi(k, w_0)$ for $\text{Im}(w_0) < 0$ we must be sure that what we determining is the analytic continuation of $\phi(k, w_0)$ for $w_0 > 0$. (Remember our method of evaluating $\phi_i(t)$ relied analytically continuing the integrand into the lower half $\omega$-plane.) The proper analytic continuation of $\phi(k, w_0)$ for $\text{Im}(w_0) < 0$
Is obtained by deforming the integration path in $V_{ii}$ so that it remains below the singularity.

If $w_0$ is real, the contour is

\[ \frac{w_0}{R} \]
Thus, for $\omega_0$ - real

$$E(k, \omega_0) = 1 + \frac{4\pi q^2}{mk^2} \left\{ P \int dV \frac{k \cdot \vec{f}_0}{\omega_0 - k \cdot V} - \frac{i\pi}{k} \frac{2}{dV} \int_{V_n = \frac{\omega_0}{k}} dV \cdot \vec{f}_0 \right\}$$

Let us assume that a solution exists for $E(\omega_0) = 0$ with $\omega_0 / k \gg V_m$

Then

$$P \int dV \frac{k \cdot \vec{f}_0}{\omega_0 - k \cdot V} = -\frac{k^2}{\omega_0^2} \int dV \cdot \vec{f}_0$$

And

$$f(V_n) = \int dV_{1} \cdot \vec{f}_0 \quad \text{is small}$$

$$\omega = 1 - \frac{\omega_p^2}{\omega^2} - \frac{i\pi \omega_p^2}{k^2} \left| \frac{\partial f(V_n)}{\partial V_n} \right|_{V_n = \frac{\omega_0}{k}} \approx E_0 + i \varepsilon_0 = 0$$
\( \varepsilon_I \ll \varepsilon_R \)

so

\[ \omega_0 = \bar{\omega}_0 + \delta \omega_0 \quad \text{where} \quad \varepsilon_R(\bar{\omega}_0) = 0 \]

with \( \delta \omega_0 \ll \bar{\omega}_0 \)

\[ 1 - \frac{\omega_p^2}{\bar{\omega}_0^2} = 0 \quad \bar{\omega}_0 = \pm \omega_p \]

\[ \frac{\partial \varepsilon_R}{\partial \omega_0} \delta \omega + \imath \varepsilon_I = 0 \]

\[ \delta \omega = -\imath \frac{\varepsilon_I}{\partial \varepsilon_R} \frac{\partial \varepsilon_R}{\partial \omega_0} = \imath \frac{\pi}{2} \frac{\omega_p^2}{k R} \frac{\partial P}{\partial v_n} \left[ \frac{\omega_p}{\bar{\omega}_0^3} \right]^{-1} \]

\[ \text{Thus if} \quad \left. \frac{R}{\bar{\omega}_0} \frac{\partial P}{\partial v_n} \right|_{v_n = \frac{30}{R}} < 0 \quad \text{THE MODE IS DAMPED} \]

\( \Phi(v_n) \)

NEGATIVE SLOPE GIVES DAMPING
Damping occurs because particles with velocity $v_{\parallel}$, such that $w_0 < k v_{\parallel}$, see a nearly time independent field. Particles which are slightly slower than the wave will be accelerated and gain energy from the wave. Particles which are slightly faster than the wave will slow down and give energy to the wave. Since there are more slower particles than faster ones ($df/dv_{\parallel} < 0$), the wave damps.
Physical Interpretation of Landau damping

Total Energy of Plasma (particles + fields) is conserved

- No collisions dissipation
- No sources

In damping process

\[ U = \frac{1}{4} E^2 \frac{\partial^2 (\omega E)}{\partial \omega^2} \]

Field Energy + Energy of Coherent motion

Plasma Waves \( E \approx 1 - \frac{w_p^2}{w^2} \)

\[ U = \frac{\partial \omega E}{\partial \omega} \] but \( E = 0 \) ?

\[ \frac{2}{\delta \omega} \omega E = \frac{2}{\delta \omega} (\omega - w_p^2) = 1 + w_p^2 \frac{1}{\omega^2} \]
Examine Resonant Particles

Electric field

\[ E(z, t) = - \frac{2}{\varepsilon_0} \Phi_0 \cos(k_\parallel z - \omega_0 t) \]

Remember, damping is weak \( \delta \omega \ll \omega_0 \)
treat \( \Phi_0 \) as constant

Transform to a frame moving with the wave

\[
\begin{align*}
Z &= Z' + \frac{\omega_0 t'}{k_\parallel} \\
V_Z &= V_{Z'} + \frac{\omega_0}{k_\parallel}
\end{align*}
\]

\( \text{Galilean} \)
\( \text{neglect relativity} \)

In the prime frame

\[ E(Z', t^\prime) = - \frac{2}{\varepsilon_0} \Phi_0 \cos(k_\parallel Z') \]

no time dependence!
Hamiltonian in this frame is conserved

\[ H' = \frac{1}{2} m V_z^2 + q \phi_o \cos k_z z' = \text{const} \quad \text{for particles} \]
Initial distribution \( f_0(V_z) \)

- Half points will increase \( V_z \)
- Half will decrease

Particle energy increased

\[
\begin{align*}
\text{Low } f & \quad \text{High } f \\
\text{Low } & \quad \text{High }
\end{align*}
\]

This will happen until roughly

\[
T = \frac{\pi}{\omega_B} - \text{bounce time}
\]

Where \( T \) is the time for a trapped particle to complete one circuit in the well.
Estimate this time by expanding around \( \bar{o} \) point

\[
\cos(k_{x}z) = 1 - \frac{1}{2}(k_{x}z)^{2}
\]

\[
H = \frac{1}{2}m\dot{v}_{x}^{2} - q\phi_{0} + g\phi_{0} \frac{1}{2}(k_{x}z - \bar{p})^{2}
\]

Harmonic Oscillator

\[
\omega_{b}^{2} = \frac{k_{x}q\phi_{0}}{m}
\]

\[
T = \frac{1}{\omega_{b}} / \sqrt{\frac{k_{x}q\phi_{0}}{m}}
\]

for small amplitude wave \( T \to \infty \)

When does linear theory apply?

1) if damping rate \( \gamma \)

\[
\gamma = -\text{Im}(\delta_{\omega})
\]

is large enough such that \( \delta T > 1 \)

wave damps before particles bounce

linear theory o.k.

otherwise nonlinear theory applies
Energy required to flatten \( f \)

\[
S_f = -(V_z - \omega) \left( \frac{\partial f}{\partial V_z} \right)_{V_z = \omega}
\]

\[
SV \sim \sqrt{\frac{2\phi_0}{m}}
\]

Charge in particle

\[
\frac{\omega}{k} + SV
\]

\[
\frac{\omega}{k} - SV
\]

\[
SU = \int \frac{1}{2} mV^2 \, SF \, d^3 \mathbf{v}
\]

\[
SU = \frac{2f}{V_z} \left[ \int_{\omega}^{\omega + SV} dV_z \left( \frac{1}{2} mV_z^2 \left( V_z - \frac{\omega}{k} \right) \right) \right]
\]

\[
= m(\omega/k) \frac{SV^2}{2}
\]

If \( SV > \frac{E^2}{8\pi} \left| \frac{2}{\omega} \right| \omega \delta \text{ linear damping occurs} \)
The Plasma Dispersion Function

$$\epsilon(k, \omega) = 1 + \frac{4\pi q^2}{mk^2} \int \frac{d^3v}{\omega - k \cdot v}$$

$$f_0 = \frac{n_0}{(\pi RT/m)^{3/2}} \exp(-\frac{1}{2}m(V_{1z}^2 + V_{2z}^2)/T)$$

Consider a Maxwellian Equilibrium Distribution function

do $v_1$ integration

$$k \cdot \frac{\partial f_0}{\partial v} = -k_0 v_z m f_0$$

$$\epsilon(k, \omega) = 1 - \frac{4\pi q^2 n_0}{\pi k^2 T} \int_{-\infty}^{\infty} \frac{dV_z}{(\pi RT/m)^{1/2}} \frac{k v_z}{\omega - k v_z} \exp\left(-\frac{1}{2}m v_z^2/T\right)$$

$$= 1 + \frac{4\pi q^2 n_0}{k^2 T} \int_{-\infty}^{\infty} \frac{dV_z}{(\pi RT/m)^{1/2}} \left[1 + \frac{\omega}{k v_z - \omega}\right] \exp\left(-\frac{1}{2}m v_z^2/T\right)$$
\[ \epsilon(k,\omega) = 1 + \frac{4\pi q^2 n_0}{k^2 T} \left[ 1 + \int_{-\infty}^{\infty} \frac{dV_z}{\sqrt{2\pi/m}} \frac{w}{kV_z - \omega} \exp\left(-\frac{1}{2}mV_z^2/kT\right) \right] \]

Normalize \( V_z \) to \( V_{4n} \)

\[ V_{4n} = (\vartheta T/m)^{1/2} \]

\( \mathcal{V} \)

Call \( \mathcal{V} = V_z/V_4 \)

Call \( \omega/kV_4 = \xi \)

\[ \epsilon(k,\omega) = 1 + \frac{4\pi q^2 n_0}{k^2 T} \left[ 1 + \xi \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{1}{x - \xi} e^{-x^2} \right] \]

The integral

\[ Z(\xi) = \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{e^{-x^2}}{x - \xi} \]

is called the plasma dispersion function

for \( \xi \) related to error function
A related function used by the some others is

\[ W(\xi) = \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{x e^{-x^2}}{\xi - x} \]

\[ 1 + \xi Z(\xi) = -W(\xi) \]

For large argument \( \xi > > 1 \)

\[ \frac{1}{x - \xi} = -\frac{1}{(\xi - x)} \approx -\frac{1}{\xi} \left[ 1 + \frac{x}{\xi} + \frac{x^2}{\xi^2} + \frac{x^3}{\xi^3} + \cdots \right] \]

\[ Z(\xi) = -\int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} e^{-x^2} \frac{1}{\xi} \left[ 1 + \frac{x}{\xi} + \frac{x^2}{\xi^2} + \frac{x^3}{\xi^3} + \cdots \right] \]

\[ = -\frac{1}{\xi} \left[ 1 + \frac{0}{\xi} + \frac{1}{2\xi^2} + \frac{0}{\xi^3} + \frac{3}{4} \frac{1}{\xi^4} \right] \]
\[ 1 + \xi Z(\xi) = 1 - \left( 1 + \frac{1}{2} \xi^2 + \frac{3}{4} \frac{1}{3} \xi^4 \right) \]

\[ \epsilon = 1 - \frac{4 \pi q^2 n_0}{k^2 T} \left[ \frac{1}{2} \frac{k^2 V_t^2}{\omega^2} + \frac{3}{4} \frac{k^4 V_t^4}{\omega^4} \right] \]

\[ \epsilon = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 + \frac{3}{2} \frac{k^2 V_t^2}{\omega^2} \right] \quad \left[ \frac{2T}{m} = V_t^2 \right] \]

\[ \epsilon = 0 \]

\[ \omega^2 = \omega_p^2 \left[ 1 + 3 \frac{k^2 T_e}{m \omega^2} \right] \]

\[ \omega \]

\[ \text{dispersion} \]

\[ k \]
PLASMA DISPERSION FUNCTION

Definition\(^{16}\) (first form valid only for \(\text{Im} \, \zeta > 0\):

\[
Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{+\infty} dt \exp \left( -\frac{t^2}{t - \zeta} \right) = 2i \exp \left( -\zeta^2 \right) \int_{-\infty}^{i\zeta} dt \exp \left( -t^2 \right).
\]

Physically, \(\zeta = x + iy\) is the ratio of wave phase velocity to thermal velocity.

Differential equation:

\[
\frac{dZ}{d\zeta} = -2(1 + \zeta Z), \quad Z(0) = i\pi^{1/2}, \quad \frac{d^2 Z}{d\zeta^2} + 2\zeta \frac{dZ}{d\zeta} + 2Z = 0.
\]

Real argument \((y = 0)\):

\[
Z(x) = \exp \left( -x^2 \right) \left( i\pi^{1/2} - 2 \int_0^x dt \exp \left( t^2 \right) \right).
\]

Imaginary argument \((x = 0)\):

\[
Z(iy) = i\pi^{1/2} \exp \left( y^2 \right) \left[ 1 - \text{erf}(y) \right].
\]

Power series (small argument):

\[
Z(\zeta) = i\pi^{1/2} \exp \left( -\zeta^2 \right) - 2\zeta \left( 1 + 2\zeta^2 / 3 + 4\zeta^4 / 15 - 8\zeta^6 / 105 + \ldots \right)
\]

Asymptotic series, \(|\zeta| \gg 1\) (Ref. 17):

\[
Z(\zeta) = i\pi^{1/2} \sigma \exp \left( -\zeta^2 \right) - \zeta^{-1} \left( 1 + 1/2\zeta^2 + 3/4\zeta^4 + 15/8\zeta^6 + \ldots \right),
\]

where

\[
\sigma = \begin{cases} 
0 & y > |x|^{-1} \\
1 & |y| < |x|^{-1} \\
2 & y < -|x|^{-1}
\end{cases}
\]

Symmetry properties (the asterisk denotes complex conjugation):

\[
Z(\zeta^*) = -[Z(-\zeta)]^*;
\]

\[
Z(\zeta^*) = [Z(\zeta)]^* + 2i\pi^{1/2} \exp[-(\zeta^*)^2] \quad (y > 0).
\]

Two-pole approximations\(^{18}\) (good for \(\zeta\) in upper half plane except when \(y < \pi^{1/2}x^2 \exp(-x^2), \quad x \gg 1\)):

\[
Z(\zeta) \approx \frac{0.50 + 0.81i}{a - \zeta} - \frac{0.50 - 0.81i}{a^* + \zeta}, \quad a = 0.51 - 0.81i;
\]

\[
Z'(\zeta) \approx \frac{0.50 + 0.96i}{(b - \zeta)^2} + \frac{0.50 - 0.96i}{(b^* + \zeta)^2}, \quad b = 0.48 - 0.91i.
\]
Imaginary Part of $Z$ for real $\xi$

$\text{Im} \{Z(\xi)\} = \pi^{\frac{1}{2}} e^{-\frac{\xi^2}{2}}$

For small argument

$Z(\xi) \approx i \pi^{\frac{1}{2}} e^{-\frac{\xi^2}{2}} - 2\xi \left(1 - 2\frac{\xi^2}{3} + \frac{4\xi^4}{15} \right)$

$1 + \frac{3}{4} Z(\xi) \approx \xi + \frac{3}{4} \xi^2$

For large argument

$1 + \frac{3}{4} Z(\xi) \approx -\left(\frac{1}{2\xi^2} + \frac{3}{4\xi^4}\right) + i \pi^{\frac{1}{2}} e^{-\frac{\xi^2}{2}}$
\[ \text{Re}\{1 + 5Z(\zeta)\} \]

Isothermal fluid

Adiabatic fluid regime

\[ \text{Im}\ 5Z(\zeta) \]
\[ e = 1 + \frac{4 \pi q^2 \rho_0}{\Gamma} \left[ 1 + \zeta \zeta'(\zeta) \right] \]

Suppose we didn't worry about \( \text{Im} \{ \zeta \} \)

\( \varepsilon_r > 0 \) has two solutions

\[ \delta n = -n_0 q \frac{\Phi}{\Gamma} \]

(highly clamped)

Suppose \( \omega \to 0 \)

"adiabatic"

iso-thermal