Spring 2018, PHYS606 - HW # 2

Problem A:

Laplacian nature of \( f(z) \):

Since \( f(z) \) is analytic, it is guaranteed that its derivative at any point \( z = x + iy \) is well defined. Hence:

\[
\lim_{\Delta \to 0} \frac{f(z + \Delta) - f(z)}{\Delta} = \lim_{\Delta \to 0} \frac{f(z + i\Delta) - f(z)}{i\Delta} \quad (1)
\]
\[
\lim_{\Delta \to 0} \frac{f(x + \Delta, y) - f(x, y)}{\Delta} = \lim_{\Delta \to 0} \frac{f(x, y + \Delta) - f(x, y)}{i\Delta}
\]
\[
\frac{\partial}{\partial x} f(x, y) = -i \frac{\partial}{\partial y} f(x, y) \quad \rightarrow \quad \partial_x f_R + i \partial_x f_I = -i \partial_y f_R + \partial_y f_I \quad (2)
\]

This leads to the well-known Cauchy-Riemann equations:

\[
\frac{\partial}{\partial x} f_R = \frac{\partial}{\partial y} f_I \quad , \quad \frac{\partial}{\partial x} f_I = -\frac{\partial}{\partial y} f_R \quad (3)
\]

Proceed to take derivatives of (2), which exist since \( f(z) \) is analytic:

\[
\frac{\partial^2}{\partial x^2} f_R = \frac{\partial^2}{\partial x \partial y} f_I \quad , \quad \frac{\partial^2}{\partial y \partial x} f_I = -\frac{\partial^2}{\partial y^2} f_R
\]
\[
\frac{\partial^2}{\partial y^2} f_R = \frac{\partial^2}{\partial y^2} f_I \quad , \quad \frac{\partial^2}{\partial x^2} f_I = -\frac{\partial^2}{\partial x \partial y} f_R
\]

Since \( f(z) \) is analytic, then \( \partial_x \partial_y = \partial_y \partial_x \):

\[
\frac{\partial^2}{\partial x^2} f_R = \frac{\partial^2}{\partial x \partial y} f_I \quad = \quad \frac{\partial^2}{\partial y \partial x} f_I = -\frac{\partial^2}{\partial y^2} f_R
\]
\[
\frac{\partial^2}{\partial y^2} f_I = \frac{\partial^2}{\partial y^2} f_R \quad = \quad \frac{\partial^2}{\partial x \partial y} f_R = -\frac{\partial^2}{\partial x^2} f_I
\]

Therefore, the real and imaginary parts of an analytic function \( f(z) \) are solutions to the Laplace equation:

\[
\nabla^2 f_R = \nabla^2 f_I = 0 \quad (4)
\]
Understanding $\arcsin z$:

Consider the function $f(z) = f(x + iy) \equiv \frac{2\phi}{\pi} \arcsin\left(\frac{z}{a}\right)$, with $X \equiv \frac{\pi}{2\phi_0} \text{Re}[f(z)]$ and $Y \equiv \frac{\pi}{2\phi_0} \text{Im}[f(z)]$. Explicitly,

$$\arcsin\left(\frac{x + iy}{a}\right) = \frac{\pi}{2\phi_0} (\text{Re}[f(z)] + i\text{Im}[f(z)]) \rightarrow \frac{x + iy}{a} = \sin(X + iY)$$

$$\frac{x + iy}{a} = \sin(X) \cos(iY) + \cos(X) \sin(iY) = \sin(X) \cosh(Y) + i \cos(X) \sinh(Y) \quad (5)$$

We are interested in the behavior of $\sin X$, so isolate $\cosh Y$ in (5):

$$\cos^2 X + \sin^2 X = 1 = \frac{x^2}{a^2 \cosh^2 Y} + \frac{y^2}{a^2 \sinh^2 Y} = \frac{x^2}{a^2 \cosh^2 Y} + \frac{y^2}{a^2 (\cosh^2 Y - 1)}$$

$$a^2 \cosh^2 Y (\cosh^2 Y - 1) = x^2 (\cosh^2 Y - 1) + y^2 \cosh^2 Y$$

$$0 = a^2 \cosh^4 Y - (x^2 + y^2 + a^2) \cosh^2 Y + x^2$$

$$\cosh^2 Y = \frac{x^2 + y^2 + a^2 \pm \sqrt{(x^2 + y^2 + a^2)^2 - 4a^2x^2}}{2a^2} \quad (6)$$

Note that for $x = y = 0$, $\cosh^2 Y = 1, 0$: as $Y \in \mathbb{R}$, then $\cosh^2 Y \geq 1$, so that only the positive root is valid.

Though correct, (6) is unwieldy, but it can be simplified using the following well-known trick (common when dealing with radicals or half-angle formulas):

$$\left(\sqrt{A} + \sqrt{B}\right)^2 = A + B + 2\sqrt{AB}$$

Identifying:

$$A + B = \frac{x^2 + y^2 + a^2}{2a^2}, \quad 2\sqrt{AB} = \frac{\sqrt{(x^2 + y^2 + a^2)^2 - 4a^2x^2}}{2a^2} \rightarrow$$

$$A + \frac{(x^2 + y^2 + a^2)^2 - 4a^2x^2}{16a^4} = \frac{x^2 + y^2 + a^2}{2a^2}$$

$$A = \frac{x^2 + y^2 + a^2}{4a^2} \pm \frac{1}{2} \sqrt{\frac{(x^2 + y^2 + a^2)^2}{4a^4} - \frac{(x^2 + y^2 + a^2)^2 - 4a^2x^2}{4a^4}} = \frac{x^2 + y^2 + a^2 \pm 2a|x|}{4a^2}$$

$$B = \frac{x^2 + y^2 + a^2}{2a^2} - \left(\frac{x^2 + y^2 + a^2 \pm 2a|x|}{4a^2}\right) = \frac{x^2 + y^2 + a^2 \mp 2a|x|}{4a^2}$$
Thus, the physical quantity $\sqrt{A} + \sqrt{B}$:

$$\sqrt{A} + \sqrt{B} = \frac{\sqrt{x^2 + y^2 + a^2 + 2a|x|}}{2a} + \frac{\sqrt{x^2 + y^2 + a^2 - 2a|x|}}{2a}$$

$$= \frac{\sqrt{(x + a)^2 + y^2}}{2a} + \frac{\sqrt{(x - a)^2 + y^2}}{2a}$$

(7)

Hence,

$$\cosh^2 Y = \left(\sqrt{A} + \sqrt{B}\right)^2$$

$$\cosh Y = \sqrt{\frac{(x + a)^2 + y^2}{a}} + \frac{\sqrt{\frac{(x - a)^2 + y^2}{a}}}{2a}$$

$$\sin X = \frac{x}{a \cosh Y} = \frac{2xa}{a} \left(\sqrt{\frac{(x + a)^2 + y^2}{a}} + \frac{\sqrt{\frac{(x - a)^2 + y^2}{a}}}{2a}\right)^{-1}$$

$$\sin X = \left(\sqrt{\frac{(x + a)^2 + y^2}{a}} - \sqrt{\frac{(x - a)^2 + y^2}{a}}\right) \frac{2x}{(x + a)^2 + y^2 - (x - a)^2 - y^2}$$

$$\Re [f(z)] = \frac{2\phi_0}{\pi} \arcsin \frac{1}{2a} \left(\sqrt{\frac{(x + a)^2 + y^2}{a}} - \sqrt{\frac{(x - a)^2 + y^2}{a}}\right)$$

Therefore, the potential representing the real part of $f(z)$ is:

$$\Re [f(z)] = \frac{2\phi_0}{\pi} \arcsin \frac{1}{2a} \left(\sqrt{\frac{(x + a)^2 + y^2}{a}} - \sqrt{\frac{(x - a)^2 + y^2}{a}}\right) \quad (8)$$

In order to gain insight into the physical interpretation of (8), observe the $x$-axis:

$$\Re [f] (x > a, y = 0) = \frac{2\phi_0}{\pi} \arcsin \frac{1}{2a} (|x + a| - |x - a|) = \frac{2\phi_0}{\pi} \arcsin (1) = \phi_0$$

$$\Re [f] (x < a, y = 0) = \frac{2\phi_0}{\pi} \arcsin \frac{1}{2a} (|x + a| - |x - a|) = \frac{2\phi_0}{\pi} \arcsin (-1) = -\phi_0$$

$$\Re [f] (|x| \leq a, y = 0) = \frac{2\phi_0}{\pi} \arcsin \frac{1}{2a} (|x + a| - |x - a|) = \frac{2\phi_0}{\pi} \arcsin \frac{x}{a} \quad (9)$$

Therefore, (8) represents any $\hat{z}$ cross-section of two semi-infinite plates, $S_+$ and $S_-$ located at $(x > a, y = 0)$ and $(x < a, y = 0)$, respectively. The plates are set in a chargeless system, held at potentials $\pm \phi_0$ for $S_\pm$, with a transitional potential given by: $\frac{2\phi_0}{\pi} \arcsin \frac{x}{a} (|x| < a, y = 0)$.
Given that the system is chargeless, φ is guaranteed continuity, hence, the existence of a transitional potential from $S_-$ to $S_+$. We approximate the exact solution by superposing individual copies of a finite plate set at some finite potential $\phi_{\text{plate}}$ on some segment, in an otherwise grounded $\tilde{x}$-axis: $S_\pm$ are long plates, and the transitional potential is modeled by a series of mini-slabs. Though, aside from this educational purpose of this setup, one could just as easily solve the Laplace equation for the relevant boundary conditions.
Problem B:

Point charge outside of a grounded conducting sphere

Consider a grounded conducting sphere of radius $R$ and centered at the origin, as well as a point charge $Q$ outside the sphere and a position $\vec{z}_0$ from the origin. Using the method of images we suggest that the contribution from the conducting sphere can be modeled by that of an imaginary charge $Q_m$ a distance $z_m$ along $\vec{z}_0$. Descriptively:

$$
\phi(\vec{r}) = \frac{1}{4\pi \epsilon_0} \left( \frac{Q}{|\vec{r} - \vec{z}_0|} + \frac{Q_m}{|\vec{r} - z_m \vec{z}_0|} \right)
$$

(10)

where $z_0 > R > z_m > 0$. If we choose appropriate $Q_m$ and $z_m$ such that $\phi(r = R) = 0$, then the solution satisfies the boundary conditions and is guaranteed to be the unique potential for the system. If $\theta$ is the angle associated with the inner product $\vec{r} \cdot \vec{z}_0$, then:

$$
|\vec{r} - \vec{z}_0| = \left[ r^2 + z_0^2 - 2r \cdot \vec{z}_0 \right]^{1/2} = \left[ r^2 + z_0^2 - 2z_0 r \cos \theta \right]^{1/2}
$$

$$
|\vec{r} - z_m \vec{z}_0| = \left[ r^2 + z_m^2 - \frac{2z_m}{z_0} r \cdot \vec{z}_0 \right]^{1/2} = \left[ r^2 + z_m^2 - 2z_m r \cos \theta \right]^{1/2}
$$

To solve for $Q_m$ and $z_m$ pick two points on the sphere’s surface, $(r = R, \theta = 0, \pi)$. Thus:

$$
0 = \frac{Q}{\sqrt{R^2 + z_0^2 - 2z_0 R}} + \frac{Q_m}{\sqrt{R^2 + z_m^2 - 2z_m R}} = \frac{Q}{|R - z_0|} + \frac{Q_m}{|R - z_m|} = \frac{Q}{z_0 - R} + \frac{Q_m}{R - z_m}
$$

$$
0 = \frac{Q}{\sqrt{R^2 + z_0^2 + 2z_0 R}} + \frac{Q_m}{\sqrt{R^2 + z_m^2 + 2z_m R}} = \frac{Q}{|R + z_0|} + \frac{Q_m}{|R + z_m|} = \frac{Q}{z_0 + R} + \frac{Q_m}{R + z_m}
$$

$$
\frac{R - z_m}{z_0 - R} = \frac{R + z_m}{z_0 + R} \Rightarrow (R - z_m)(z_0 + R) = (R + z_m)(z_0 - R) \Rightarrow R^2 = z_m z_0 \quad \text{(11)}
$$

$$
\frac{Q_m}{Q} = \frac{z_m - R}{z_0 - R} = \frac{\frac{R^2}{z_0} - R}{z_0 - R} \Rightarrow Q_m = -\frac{RQ}{z_0} \quad \text{(12)}
$$

Thus, (10) becomes:

$$
\phi(\vec{r}) = \frac{Q}{4\pi \epsilon_0} \left( \frac{1}{\sqrt{r^2 + z_0^2 - 2z_0 r \cos \theta}} - \frac{1}{\sqrt{R^2 + \frac{z_m^2 r^2}{R^2} - 2z_0 r \cos \theta}} \right)
$$

(13)

Note that for $r = R$ the denominators in (13) are the same, thus corroborating $\phi(r = R) = 0$. 


### Point charge outside of an isolated conducting sphere

An isolated conducting sphere (in the absence of other charges) has a constant potential $V_0$ throughout. Outside of the sphere, the potential is that of a charge $Q_0$ at the center of the sphere (where $Q_0 = 4\pi \varepsilon_0 a V_0$). In the presence of another charge $Q$, the grounded solution from (10) holds, with an additional image charge $Q_{00} = Q_0 - Q_m$ located at the center of the sphere. This additional term accounts for the fixed total charge of the isolated sphere:

\[
\phi (\vec{r}) = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q}{|\vec{r} - \vec{z}_0|} + \frac{Q_m}{|\vec{r} - z_m \vec{z}_0|} + \frac{Q_{00}}{|\vec{r}|} \right) \quad (14)
\]

\[
\phi (\vec{r}) = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q}{|\vec{r} - \vec{z}_0|} + \frac{-\frac{Q}{z_0}}{\left(z_0 - R^2\right)^2} + \frac{Q_0 + \frac{RQ}{z_0}}{z_0^2} \right) \quad (15)
\]

Exploiting the method of images, we obtain the force $\vec{F}$ of the sphere acting on $Q$ by considering pair-wise point interactions between $Q$ and the image charges. Note that all charges are placed in $\vec{z}_0$ so that $\vec{F} = F \hat{z}_0$, thus:

\[
F = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{-\frac{RQ}{z_0}}{\left(z_0 - R^2\right)^2} + \frac{Q_0 + \frac{RQ}{z_0}}{z_0^2} \right] = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{-RzQ}{(z_0 - R^2)^2} + \frac{Q_0}{z_0^2} + \frac{RQ}{z_0^3} \right] \quad (16)
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \left[ RQ \left( \frac{1}{z_0^3} - \frac{z_0}{(z_0^2 - R^2)^2} \right) + \frac{Q_0}{z_0^2} \right] = \frac{Q}{4\pi \varepsilon_0} \left[ RQ \left( \frac{z_0^4 - 2z_0^2R^2 + R^4 - z_0^2}{z_0^3(z_0^2 - R^2)^2} \right) + \frac{Q_0}{z_0^2} \right]
\]

Hence, the force acting on $Q$ is:

\[
\vec{F} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{Q_0}{z_0^2} - \frac{R^3Q}{z_0^3(z_0^2 - R^2)^2} \right] \vec{z}_0 \quad (17)
\]

This result (17) is valid for any point $z_0$; hence, integrate about an arbitrary $z$ to obtain the total work:

\[
W = \int d\ell \cdot \vec{F} = \int_{-\infty}^{z_0} \frac{Q}{4\pi \varepsilon_0} \left[ \frac{-RzQ}{(z^2 - R^2)^2} + \frac{Q_0}{z^2} + \frac{RQ}{z^3} \right] = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{RQ}{2(z^2 - R^2)} - \frac{Q_0}{z} - \frac{RQ}{2z^2} \right]_{-\infty}^{z_0}
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \left[ \frac{RQ}{2(z_0^2 - R^2)} - \frac{Q_0}{z_0} - \frac{RQ}{2z_0^2} \right] = \frac{Q}{4\pi \varepsilon_0} \left[ -\frac{Q_0}{z_0} + \frac{RQ}{2} \left( \frac{1}{z_0^2 - R^2} - \frac{1}{z_0^2} \right) \right]
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \left[ -\frac{Q_0}{z_0} + \frac{R^3Q}{2z_0^2(z_0^2 - R^2)} \right]
\]
Therefore, the work necessary to construct the system is given by:

\[ W = \frac{Q}{4\pi \varepsilon_0} \left[ -\frac{Q_0}{z_0} + \frac{R^3 Q}{2z_0^2 (z_0^2 - R^2)} \right] \] (18)

Note that the first term, \( W_0 = -\frac{QQ_0}{4\pi \varepsilon_0 z_0^3} \) is the work necessary to arrange the \( Q-Q_0 \) configuration, which we expect from Coulomb’s Law. For the second term, \( W_c = \frac{R^3 Q^2}{8\pi \varepsilon_0 z_0^3 (z_0^2 - R^2)} \), \( W_c > 0 \), and it vanishes for \( R \ll z_0 \) (as expected). Interestingly, \( W_c > 0 \) indicates an attractive contribution between \( Q \) and \( Q_m \), which is consistent with our result of: \( \text{sgn} (Q) = -\text{sgn} (Q_m) \).

**Dipole outside of a grounded conducting sphere**

Consider a grounded conducting sphere of radius \( R \) and centered at the origin, as well as a dipole \( \vec{p} \) outside the sphere and a position \( z_0 \) from the origin. We model the dipole as two charges \( \pm Q \) located at \( \vec{R}_\pm = z_0 \pm \Delta \hat{p} \), for \( R \gg \Delta > 0 \). A direct calculation of the modeled dipole reveals:

\[ \vec{p} = Q \left( \vec{R}_+ - z_0 \right) + (-Q) \left( \vec{R}_- - z_0 \right) = Q \Delta \hat{p} - (-Q) \Delta \hat{p} = 2Q \Delta \hat{p} \quad \rightarrow \quad p = 2Q \Delta \] (19)

According to (10 - 13) the charges \( \pm Q \) have mirror images \( q_\mp \) located at \( \vec{w}_\pm \):

\[ q_\mp = \mp \frac{RQ}{R_\pm} \quad \text{and} \quad \vec{w}_\pm = \frac{R^2}{R_\pm} \vec{R}_\pm = \frac{R^2}{R_\pm} (z_0 \pm \Delta \hat{p}) \] (20)

Since we are working with dipoles, expansions to the first order in \( \Delta \) are acceptable. Thus, consider the following derivative:

\[ \frac{\partial R_\pm}{\partial \Delta} = \frac{\partial}{\partial \Delta} \sqrt{z_0^2 + \Delta^2 \pm 2 \Delta z_0 \cdot \hat{p}} = \frac{1}{2} \left[ z_0^2 + \Delta^2 \pm 2 \Delta z_0 \cdot \hat{p} \right]^{-1/2} \left( 2 \Delta \pm 2 z_0 \cdot \hat{p} \right) = \frac{\Delta \pm z_0 \cdot \hat{p}}{R_\pm} \] (21)

Expand (20) to first order in \( \Delta \) using (21):

\[ q_\mp = \mp RQ \left[ \frac{1}{z_0} \pm \left( -\frac{1}{R_\pm^2} \times \frac{\Delta \pm z_0 \cdot \hat{p}}{R_\pm} \right) \right] \Delta = \mp RQ \left[ \frac{1}{z_0} \pm \frac{z_0 \cdot \hat{p}}{z_0^3} \Delta \right] = RQ \left[ \frac{z_0 \cdot \hat{p}}{z_0^3} \Delta \pm \frac{1}{z_0} \right] \] (22)

\[ \vec{w}_\pm = R^2 \left( z_0 \pm \Delta \hat{p} \right) \left[ \frac{1}{z_0^2} \pm \left( -\frac{2}{R_\pm^2} \times \frac{\Delta \pm z_0 \cdot \hat{p}}{R_\pm} \right) \right] \Delta = R^2 \left( z_0 \pm \Delta \hat{p} \right) \left[ \frac{1}{z_0^2} \pm \frac{2z_0 \cdot \hat{p}}{z_0^3} \Delta \right] \] (23)

Note that (23) contains a remarkable amount of physics: \( \lim_{\Delta \to 0} \vec{w}_\pm = \frac{R^2}{z_0^2} z_0 \), and \( \left| \vec{w}_+ - \frac{R^2}{z_0^2} z_0 \right| = \left| \vec{w}_- - \frac{R^2}{z_0^2} z_0 \right| \).

This suggests that at least geometrically, we can attempt to define a mirror-dipole \( \vec{p}_m \) located at \( \frac{R^2}{z_0^2} z_0 \), as long as we give some consideration to the corresponding charge. In general \( |q_+| \neq |q_-| \), so we proceed to break the charges into three parts: two charges \( \pm q_m \) associated with \( \vec{p}_m \), and a charge \( Q_m \) corresponding...
to a point charge, to wit:

\[ Q_m = \text{sgn} \left( \vec{z}_0 \cdot \hat{p} \right) \left\{ \max (|q_-|, |q_+|) - \min (|q_-|, |q_+|) \right\} \]

\[ = \text{sgn} \left( \vec{z}_0 \cdot \hat{p} \right) \left\{ R_Q \left[ \frac{1}{z_0} + \frac{|\vec{z}_0 \cdot \hat{p}|}{z_0^3} \Delta \right] - R_Q \left[ \frac{1}{z_0} - \frac{|\vec{z}_0 \cdot \hat{p}|}{z_0^3} \Delta \right] \right\} \]

\[ = \frac{2RQ\Delta}{z_0^2} |\vec{z}_0 \cdot \hat{p}| \text{sgn} (\vec{z}_0 \cdot \hat{p}) = \frac{Rp}{z_0^2} (\vec{z}_0 \cdot \hat{p}) = \frac{R}{z_0^2} (\vec{z}_0 \cdot \hat{p}) \]

(24)

\[ \pm q_m = \pm \min (|q_-|, |q_+|) = \pm RQ \left[ \frac{1}{z_0} - \frac{|\vec{z}_0 \cdot \hat{p}|}{z_0^3} \Delta \right] \]

(25)

in which we exploited (19) for (24) as all \( \Delta \) effects are accounted in \( Q_m \). Having an explicit form for \( \pm q_m \), it is straightforward to compute \( \vec{p}_m \):

\[ \vec{p}_m = (-q_m) \left( \vec{w}_+ - \frac{R^2}{z_0^2} \vec{z}_0 \right) + (q_m) \left( \vec{w}_- - \frac{R^2}{z_0^2} \vec{z}_0 \right) \]

\[ = -q_m \left( \hat{p} - \frac{2(\vec{z}_0 \cdot \hat{p})}{z_0^2} \right) R^2 \frac{\Delta}{z_0^2} + q_m (-1) \left( \hat{p} - \frac{2(\vec{z}_0 \cdot \hat{p})}{z_0^2} \right) R^2 \frac{\Delta}{z_0^2} \]

\[ = -2RQ \left[ \frac{1}{z_0} - \frac{|\vec{z}_0 \cdot \hat{p}|}{z_0^3} \Delta \right] \left( \hat{p} - \frac{2(\vec{z}_0 \cdot \hat{p})}{z_0^2} \right) R^2 \frac{\Delta}{z_0^2} = \left( \frac{2(\vec{z}_0 \cdot \hat{p})}{z_0^2} \right) R^3 Q \frac{\Delta}{z_0^3} + o(\Delta^2) \]

(26)

in which we exploited (19) for (26) as all \( \Delta \) effects are accounted in \( \vec{p}_m \).

The potential for the full system is then:

\[ \phi (\vec{r}) = \frac{1}{4\pi \epsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{z}_0)}{|\vec{r} - \vec{z}_0|^3} + \frac{1}{4\pi \epsilon_0} \frac{\vec{p}_m \cdot (\vec{r} - \frac{R^2}{z_0} \vec{z}_0)}{|\vec{r} - \frac{R^2}{z_0} \vec{z}_0|^3} + \frac{1}{4\pi \epsilon_0} \frac{Q_m}{|\vec{r} - \frac{R^2}{z_0} \vec{z}_0|^3} \]

(27)

with \( \vec{p}_m \) defined in (26), and \( Q_m \) defined in (24).
Dipole outside of an isolated conducting sphere

Similar considerations to the case of an isolated conducting sphere under the presence of a point charge, justify adding a point-like potential at the center of the sphere. In this case, the effective charge of this potential would be \( Q_{00} = Q_0 - Q_m \), the left over charge from the dipole expansion:

\[
\phi (\vec{r}) = \frac{1}{4\pi \epsilon_0} \left( \vec{p} \cdot (\vec{r} - \vec{z}_0) \right) + \frac{1}{4\pi \epsilon_0} \left( \vec{p}_m \cdot \left( \vec{r} - \frac{\ell^2}{z_0} \vec{z}_0 \right) \right) + \frac{1}{4\pi \epsilon_0} \frac{Q_m}{|\vec{r} - \frac{\ell^2}{z_0} \vec{z}_0|} + \frac{1}{4\pi \epsilon_0} \frac{Q_{00}}{|\vec{r}|}
\]

\[
\phi (\vec{r}) = \frac{1}{4\pi \epsilon_0} \left( \vec{p} \cdot (\vec{r} - \vec{z}_0) \right) + \frac{1}{4\pi \epsilon_0} \left( \vec{p}_m \cdot \left( \vec{r} - \frac{\ell^2}{z_0} \vec{z}_0 \right) \right) + \frac{1}{4\pi \epsilon_0} \frac{Q_m}{|\vec{r} - \frac{\ell^2}{z_0} \vec{z}_0|} + \frac{1}{4\pi \epsilon_0} \frac{Q_0 - Q_m}{|\vec{r}|} \tag{28}
\]

The forces acting on the dipole \( \vec{p} \) include charge-dipole forces, and dipole-dipole forces: thus, we need explicit forms for said expressions. The force acting on a dipole \( \vec{p} \) at a position \( \ell \) from a charge \( Q \) (or a dipole \( \vec{w} \)) at the origin is given by:

\[
\vec{F}_Q = -\frac{Q}{4\pi \epsilon_0} \left( \frac{3 (\vec{p} \cdot \hat{\ell}) \hat{\ell} - \vec{p}}{\ell^3} \right) \quad \text{and} \quad \vec{F}_w = \frac{3}{4\pi \epsilon_0} \left[ \frac{\vec{w} (\vec{p} \cdot \hat{\ell}) + \hat{\ell} (\vec{p} \cdot \vec{w}) - 5 (\vec{w} \cdot \hat{\ell}) (\vec{p} \cdot \hat{\ell})}{\ell^4} \right]
\]

While \( \vec{F}_w \) does not seem self-evident, note that the derivation of the expression is tedious rather than complicated, and it follows directly from \( \vec{F}_Q \). We are interested in attractive (or repulsive) components of the force \( \vec{F} \) on \( \vec{p} \), so consider only the component along \( \hat{\ell} \):

\[
\vec{F}_Q \cdot \hat{\ell} = -\frac{Q}{4\pi \epsilon_0} \left( \frac{2 \vec{p} \cdot \hat{\ell}}{\ell^3} \right) \quad \text{and} \quad \vec{F}_w \cdot \hat{\ell} = \frac{3}{4\pi \epsilon_0} \left[ \vec{p} \cdot \vec{w} - 3 (\vec{w} \cdot \hat{\ell}) (\vec{p} \cdot \hat{\ell}) \right] \frac{1}{\ell^4}
\]

Since all our charges and dipoles are co-linear and \( z_0 > \frac{R^2}{z_0} > 0, \hat{\ell} = \hat{z}_0 \) for all force elements:

\[
F_z = \vec{F}_z \cdot z_0 = -\frac{Q_m}{4\pi \epsilon_0} \left( z_0 - \frac{R^2}{z_0} \right)^3 - \frac{Q_0 - Q_m}{4\pi \epsilon_0} \left( \frac{2 \vec{p} \cdot \hat{z}_0}{z_0} \right)^3 + \frac{3}{4\pi \epsilon_0} \left[ \vec{p} \cdot \hat{z}_0 \right. \\
&= \left. \frac{1}{4\pi \epsilon_0} \left( z_0^3 - \frac{R^2 z_0}{z_0} \right)^3 + \frac{3}{4\pi \epsilon_0} \left[ \vec{p} \cdot \hat{z}_0 \right. \\
&= \left. \frac{3}{4\pi \epsilon_0} \left( \frac{2 z_0^3}{z_0^2 - R^2} - \frac{2}{z_0^3} \right) - 2 \frac{Q_m}{4\pi \epsilon_0} \left( \frac{\ell^2}{z_0} \right)^3 + \frac{3}{4\pi \epsilon_0} \left[ \vec{p} \cdot \hat{z}_0 \right. \\
&= \left. \frac{3}{4\pi \epsilon_0} \left( \frac{R^3}{z_0^3} (2 \vec{p} \cdot \hat{z}_0)^2 - \frac{R^6}{z_0^4} \right) - 3 \frac{R^3}{z_0^4} \left( \vec{p} \cdot \hat{z}_0 \right) \right]
\]

9
\[ F_z = F_{z0} - \frac{Q_m (\vec{p} \cdot \hat{z}_0)}{4\pi \epsilon_0} \left[ \frac{2z_0^3}{(z_0^2 - R^2)^3} - \frac{2}{z_0^3} \right] + \frac{3}{4\pi \epsilon_0} \left( \frac{R^3 z_0^3}{z_0^3} \right) \left[ -\left( \hat{z}_0 \cdot \vec{p} \right)^2 - p^2 \right] \]

\[ = F_{z0} - \frac{Q_m (\vec{p} \cdot \hat{z}_0)}{4\pi \epsilon_0} \left[ \frac{2z_0^3}{(z_0^2 - R^2)^3} - \frac{2}{z_0^3} + \frac{3R^2 z_0^4}{z_0 (z_0^2 - R^2)^4} \right] - \frac{3}{4\pi \epsilon_0} \frac{R^3 z_0 p^2}{(z_0^2 - R^2)^4} \]

\[ = F_{z0} - \frac{R (\vec{p} \cdot \hat{z}_0)^2}{4\pi \epsilon_0 z_0^2} \left[ \frac{2z_0^3}{(z_0^2 - R^2)^3} - \frac{2}{z_0^3} + \frac{3R^2 z_0^3}{(z_0^2 - R^2)^4} \right] - \frac{3}{4\pi \epsilon_0} \frac{R^3 z_0 p^2}{(z_0^2 - R^2)^4} \]

Therefore, the force along \( \hat{z}_0 \) on the dipole \( \vec{p} \) is given by:

\[ F_z = F_{z0} - \frac{R (\vec{p} \cdot \hat{z}_0)^2}{4\pi \epsilon_0 z_0^2} \left[ \frac{2z_0^3}{(z_0^2 - R^2)^3} - \frac{2}{z_0^3} + \frac{3R^2 z_0^3}{(z_0^2 - R^2)^4} \right] - \frac{3}{4\pi \epsilon_0} \frac{R^3 z_0 p^2}{(z_0^2 - R^2)^4} \]  (29)

where \( F_{z0} = -\frac{Q_0}{4\pi \epsilon_0 \frac{(2R \cdot \hat{z}_0)}{z_0^3}} \), is the \( \hat{z}_0 \) component of the force acting on \( \vec{p} \) by charge \( Q_0 \) from the isolated sphere. Note that regardless of the direction of \( \vec{p} \), \( F_z - F_{z0} < 0 \), which means the force is purely attractive. Additionally, for \( \vec{p} \cdot \hat{z}_0 = 0 \), \( F_z \) remains attractive, as \( F_z - F_{z0} \propto -p^2 \) (which is incidentally, purely a dipole-dipole contribution).
The forces can be evaluated explicitly (to first order in $\Delta$):

$$\vec{F} = \frac{3}{r^3} \left[ \frac{3}{\ell^2} \left[ \vec{w} \cdot \vec{\ell} \right] + \Delta \left( \hat{p} \left( \vec{w} \cdot \vec{\ell} \right) - 5 \left( \vec{w} \cdot \vec{\ell} \right) \hat{\ell} + \vec{\ell} (\vec{w} \cdot \hat{\ell}) \right) - \hat{w} \left( 1 + \frac{3\vec{\ell} \cdot \hat{p}}{\ell^2} \right) \right]$$

The forces can be evaluated explicitly (to first order in $\Delta$):

$$\vec{F}_{\pm} = \pm \frac{Q}{4\pi\varepsilon_0} \left[ 3 \left( \vec{w} \cdot \vec{\ell}_{\pm} \right) \vec{\ell}_{\pm} \right]$$

$$\vec{F} = \vec{F}_+ + \vec{F}_- = \frac{2Q}{4\pi\varepsilon_0 \ell^3} \left[ 3\Delta \left( \hat{p} \left( \vec{w} \cdot \vec{\ell} \right) - 5 \left( \vec{w} \cdot \vec{\ell} \right) \hat{\ell} + \vec{\ell} (\vec{w} \cdot \hat{\ell}) \right) - \hat{w} \left( 1 + \frac{3\vec{\ell} \cdot \hat{p}}{\ell^2} \right) \right]$$

$$= \frac{6Q\Delta}{4\pi\varepsilon_0 \ell^4} \left[ \hat{p} \left( \vec{w} \cdot \vec{\ell} \right) - 5 \left( \vec{w} \cdot \vec{\ell} \right) \hat{\ell} + \vec{\ell} (\vec{w} \cdot \hat{\ell}) \right]$$

$$= \frac{3p}{4\pi\varepsilon_0 \ell^4} \left[ \hat{p} \left( \vec{w} \cdot \vec{\ell} \right) + \hat{\ell} (\vec{w} \cdot \hat{\ell}) - 5 \left( \vec{w} \cdot \vec{\ell} \right) \hat{\ell} \right]$$
Therefore, the force $\vec{F}$ on $\vec{p}$ is given by:

$$\vec{F} = \frac{3}{4\pi\varepsilon_0} \left[ \vec{p} (\vec{w} \cdot \hat{\ell}) + \vec{w} (\vec{p} \cdot \hat{\ell}) + \hat{\ell} (\vec{w} \cdot \vec{p}) - 5 \left( \vec{w} \cdot \hat{\ell} \right) (\vec{p} \cdot \hat{\ell}) \right] \ell^4$$ (31)

**Problem C:**

**Line charge above conducting plane**

Consider an infinite line charge $\lambda$ located at $\vec{r}_+ = r_0 (\cos \theta_0, \sin \theta_0)$, where $r_0 > 0$, $\theta_0$ is measured from $\hat{x}$ and the $\hat{x} - \hat{y}$ plane corresponds to any cross-section of this cylindrical system. Using the method of images, we obtain the potential by introducing an imaginary line charge $-\lambda$ located at: $\vec{r}_- = r_0 (\cos \theta_0, -\sin \theta_0)$. The full solution for the potential is then:

$$\phi = \frac{\lambda}{2\pi\varepsilon_0} \log \frac{r_{org}}{|\vec{r} - \vec{r}_+|} - \frac{\lambda}{2\pi\varepsilon_0} \log \frac{r_{org}}{|\vec{r} - \vec{r}_-|} = \frac{\lambda}{2\pi\varepsilon_0} \log \frac{|\vec{r} - \vec{r}_-|}{|\vec{r} - \vec{r}_+|}$$ (32)

where $r_{org}$ is an arbitrary (but finite) reference radius. Evaluate the radial component directly:

$$|\vec{r} - \vec{r}_\pm|^2 = (r \cos \theta - r_0 \cos \theta_0)^2 + (r \sin \theta \mp r_0 \sin \theta_0)^2 = r^2 + r_0^2 - 2rr_0 \cos \theta \cos \theta_0 \mp 2rr_0 \sin \theta \sin \theta_0$$

$$= r^2 + r_0^2 - 2rr_0 \cos \theta \cos \theta_0 \pm 2rr_0 \sin \theta \sin \theta_0 = r^2 + r_0^2 - 2rr_0 \cos (\theta \mp \theta_0)$$ (33)

Proceed to simplify (32) via (33):

$$\phi = \frac{\lambda}{2\pi\varepsilon_0} \log \frac{|\vec{r} - \vec{r}_-|}{|\vec{r} - \vec{r}_+|} = \frac{\lambda}{4\pi\varepsilon_0} \log \frac{r^2 + r_0^2 - 2rr_0 \cos (\theta - \theta_0)}{r^2 + r_0^2 - 2rr_0 \cos (\theta + \theta_0)}$$ (34)

where it is easy to check $\phi (\theta = 0) = \phi (\theta = \pi) = 0$. A series solution for this potential can be obtained from the Poisson equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = -\frac{\lambda}{\varepsilon_0 r_0} \delta (r - r_0) \delta (\theta - \theta_0)$$ (35)

We expect that the basis functions will contain sinusoidal behavior on $\theta$ and Bessel-like behavior on $r$. However, note that Bessel functions often have the form $J_m \left( \frac{k_m r}{L} \right)$, were $k_m$ is the $m$-th root of the $m$-th Bessel function, and $L$ is the finite length for a reference cylinder on which $\phi (r = L) = 0$. In principle we could obtain solutions using Bessel functions for some arbitrary $L$ and then take the limit as $L \to \infty$; that is however, considered a form of abuse in most modern democracies. Alternatively, note that the Laplacian on cylindrical coordinates is power-like, so consider the basis functions: $\phi_m = r^m \sin m\theta$. Note that the boundary condition $\phi (\theta = 0) = \phi (\theta = \pi) = 0$ is the reason for our choice of sinusoidal basis, and angular dependence. Applying the Laplacian operator on $\phi_m$:

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi_m}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi_m}{\partial \theta^2} = \mu_m^2 r^{m-2} \sin m\theta - m^2 \frac{\phi_m}{r^2} = \left( \mu_m^2 - m^2 \right) \frac{\phi_m}{r^2} \to \mu_m = \pm m$$ (36)
The charge density in (35) sets two different domains. For \( r < r_0 \), \( \phi \) cannot diverge at the origin, whereas \( r > r_0 \) requires \( \phi \) to vanish at infinity. Hence, the full solution for \( \phi \) is given by:

\[
\phi(r, \theta) = \sum_{m=1}^{\infty} \sin(m\theta) \begin{cases} 
A_m r^m, & r \leq r_0 \\
B_m r^{-m}, & r > r_0 
\end{cases}
\]  (37)

Imposing continuity on \( \phi(r = r_0) \) gives the constraint: \( A_m r^m_0 = B_m r^{-m}_0 \). Hence,

\[
\phi(r, \theta) = \sum_{m=1}^{\infty} A_m S_m(r) \sin(m\theta) = \sum_{m=1}^{\infty} R_m(r) \sin(m\theta) = \sum_{m=1}^{\infty} A_m \sin(m\theta) \begin{cases} 
\frac{r^m}{r_0^m}, & r \leq r_0 \\
\frac{r_0^m r^{-m}}{r_0^m r^{-m}}, & r > r_0 
\end{cases}
\]  (38)

In order to obtain \( A_m \) it is necessary to directly manipulate \( \phi \) term-wise in the Poisson equation:

\[
\sum_{m=1}^{\infty} \sin(m\theta) \left[ \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) - \frac{m^2}{r} \phi \right] = -\frac{\lambda r}{\varepsilon_0 r_0} \delta(r - r_0) \delta(\theta - \theta_0)
\]

\[
\sum_{m=1}^{\infty} \int_0^\pi d\theta \sin(n\theta) \sin(m\theta) \left[ \frac{\partial}{\partial r} \left( r \frac{\partial R_m}{\partial r} \right) - \frac{m^2}{r} R_m \right] = -\frac{\lambda r}{\varepsilon_0 r_0} \delta(r - r_0) \int_0^\pi d\theta \sin(n\theta) \delta(\theta - \theta_0)
\]

\[
\sum_{m=1}^{\infty} \frac{\pi}{2} \delta_{mn} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial R_n}{\partial r} \right) - \frac{n^2}{r} R_n \right] = -\frac{\lambda r}{\varepsilon_0 r_0} \delta(r - r_0) \sin(n\theta_0)
\]

\[
\int_{r_0 - \epsilon}^{r_0 + \epsilon} dr \left\{ \frac{\partial}{\partial r} \left( r \frac{\partial R_n}{\partial r} \right) - \frac{n^2}{r} R_n \right\} = -\frac{2\lambda r}{\pi \varepsilon_0} \sin(n\theta_0) \quad \text{(for small } \epsilon)\]

\[
\left. r_0 \frac{\partial R_n}{\partial r} \right|_{r_0 - \epsilon}^{r_0 + \epsilon} = r_0 \left. \frac{\partial R_n}{\partial r} \right|_{r_0 - \epsilon} - n^2 \int_{r_0 - \epsilon}^{r_0 + \epsilon} dr \frac{R_n}{r} = -\frac{2\lambda}{\pi \varepsilon_0} \sin(n\theta_0) \quad \text{(for small } \epsilon)\]  (39)

Using the notation from (38):

\[
A_n = \frac{2\lambda}{\pi \varepsilon_0 r_0} \sin(n\theta_0) \left[ \left. \frac{\partial S_n}{\partial r} \right|_{r=r_0-\epsilon} - \left. \frac{\partial S_n}{\partial r} \right|_{r=r_0+\epsilon} \right]^{-1}
\]  (40)

In order to evaluate \( A_m \) compute \( S_m \) directly:

\[
S_m = \begin{cases} 
\frac{r^m}{r_0^m r^{-m}}, & r \leq r_0 \\
\frac{r_0^m r^{-m}}{r_0^m r^{-m}}, & r > r_0
\end{cases}
\]

\[
A_m = \frac{2\lambda}{\pi \varepsilon_0 r_0} \sin(m\theta_0) \left[ m r_0^{m-1} - (-m) r_0^{m-1} \right]^{-1} = \frac{\lambda}{m \pi \varepsilon_0 r_0^m} \sin(m\theta_0)
\]  (41)
Therefore, \( \phi \) for a line charge on a conducting grounded plane:

\[
\phi (r, \theta) = \frac{\lambda}{\pi \varepsilon_0} \sum_{m=1}^{\infty} \frac{\sin (m\theta) \sin (m\theta_0)}{m} \left\{ \left( \frac{r}{r_0} \right)^m, r \leq r_0 \right\}
\]

\[
\left\{ \left( \frac{r_0}{r} \right)^m, r > r_0 \right\}
\]

(42)

**Line charge above conducting plane with hemisphere**

Though we still satisfy the Poisson equation in (35), there are different boundary conditions, namely, \( r \geq a \).

Therefore, the decaying contributions for \( a \leq r \leq r_0 \) do not vanish. Hence,

\[
\phi (r, \theta) = \sum_{m=1}^{\infty} \sin (m\theta) \left\{ A_m r^m + C_m r^{-m} \right\}, a \leq r \leq r_0
\]

\[
B_m r^{-m}, r > r_0
\]

(43)

Imposing \( \phi (r = a) = 0 \), and continuity at \( r = r_0 \):

\[
A_m a^m + C_m a^{-m} = 0, \quad A_m r_0^m + C_m r_0^{-m} = B_m r_0^{-m}
\]

\[
C_m = -A_m a^{2m}, \quad B_m = A_m (r_0^{-2m} - a^{2m})
\]

(44)

Therefore,

\[
\phi (r, \theta) = \sum_{m=1}^{\infty} A_m \sin (m\theta) \left\{ r^m - \left( \frac{a^2}{r} \right)^m \right\}, a \leq r \leq r_0
\]

\[
\left\{ r_0^m \left( \frac{a}{r} \right)^m - \left( \frac{a^2}{r} \right)^m \right\}, r > r_0
\]

(45)

Since the Poisson equation in (35) remains valid, and the new boundary conditions do not affect \( r = r_0 \), then (40) holds:

\[
\frac{\partial S_m}{\partial r} = \left\{ mr_0^{m-1} - ma^{2m} - r_0^{m-1} + ma^{2m} \right\}, r \leq r_0
\]

\[
- m f_0^{m-1} - ma^{2m} - r_0^{m-1} + ma^{2m} \right\}, r > r_0
\]

\[
A_m = \frac{2\lambda}{\pi \varepsilon_0 r_0} \sin (m\theta_0) \left[ m r_0^{m-1} - (ma^{2m}) r_0^{m-1} + 0 \right]^{-1} = \frac{\lambda}{m \pi \varepsilon_0 r_0^{m-1}} \sin (m\theta_0)
\]

(46)

Thus,

\[
\phi (r, \theta) = \frac{\lambda}{\pi \varepsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \sin (m\theta_0) \sin (m\theta) \left\{ \left( \frac{r}{r_0} \right)^m - \left( \frac{a^2}{r} \right)^m \right\}, a \leq r \leq r_0
\]

\[
\left\{ \left( \frac{r_0}{r} \right)^m - \left( \frac{a^2}{r} \right)^m \right\}, r > r_0
\]

(47)

for \( w_0 \equiv \frac{a^2}{r_0} \).
Charge considerations on hemisphere

Compute the linear charge density on the hemisphere directly, by sampling the electric field perpendicular to the boundary:

\[
\sigma = \epsilon_0 E_r \bigg|_{r=a} = -\epsilon_0 \frac{\partial \phi}{\partial r} \bigg|_{r=a} = -\lambda \frac{\sum_{m=1}^{\infty} \frac{1}{m} \sin (m\theta_0) \sin (m\theta) \left[ \frac{mr^{m-1}}{r_0^{m}} + m \frac{w_0^m}{r^{m+1}} \right]}{\pi r} \\
= -\frac{\lambda}{\pi r} \sum_{m=1}^{\infty} \sin (m\theta_0) \sin (m\theta) \left[ \frac{r^m}{r_0^m} + \frac{w_0^m}{r^m} \right] \bigg|_{r=a} = -\frac{2\lambda}{\pi a} \sum_{m=1}^{\infty} \sin (m\theta_0) \sin (m\theta) \left( \frac{a}{r_0} \right)^m \tag{48}
\]

Integrate over all angles to obtain the total charge at the boundary:

\[
Q = -\frac{2\lambda}{\pi a} \sum_{m=1}^{\infty} \sin (m\theta_0) \left( \int_0^\pi d\theta \sin (m\theta) \right) \left( \frac{a}{r_0} \right)^m = -\frac{2\lambda}{\pi a} \sum_{m=1}^{\infty} \sin (m\theta_0) 2 \left( \frac{\delta_{\text{odd},m}}{m} \right) \left( \frac{a}{r_0} \right)^m \\
= -\frac{4\lambda}{\pi a} \sum_{m=\text{odd}}^{\infty} \sin (m\theta_0) \left( \frac{a}{r_0} \right)^m \tag{49}
\]

Therefore, the linear charge density and total charge in the hemisphere are given by:

\[
\sigma = -\frac{2\lambda}{\pi a} \sum_{m=1}^{\infty} \sin (m\theta_0) \sin (m\theta) \left( \frac{a}{r_0} \right)^m \\
Q = -\frac{4\lambda}{\pi a} \sum_{m=\text{odd}}^{\infty} \sin (m\theta_0) \left( \frac{a}{r_0} \right)^m \tag{50}
\]

Electric field due to hemisphere

From (50) we can extract the electric field at the peak of a small hemisphere (the $\hat{\theta}$ component vanishes for the boundary) for a line charge at $\theta = \pi/2$. This entails evaluating for $\theta = \theta_0 = \pi/2$ and to leading order in $\frac{a}{r_0} \ll 1$:

\[
E_r = \frac{\sigma}{\epsilon_0} = -\frac{2\lambda}{\pi a \epsilon_0} \sum_{m=1}^{\infty} \sin (m\theta_0) \sin (m\theta) \left( \frac{a}{r_0} \right)^m = -\frac{2\lambda}{\pi a \epsilon_0} \sum_{m=1}^{\infty} \delta_{\text{odd},m} \left( \frac{a}{r_0} \right)^m = -\frac{2\lambda}{\pi r_0 \epsilon_0} \tag{51}
\]
In the absence of a hemisphere, the electric field can be obtained directly from (42):

\[ E_{0, \theta} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\lambda}{\pi \epsilon_0 r} \sum_{m=1}^{\infty} \sin (m\theta) \cos (m\theta_0) \left( \frac{r}{r_0} \right)^m \]

\[ = -\frac{\lambda}{\pi \epsilon_0 a} \sum_{m=1}^{\infty} \frac{\sin (m(\theta + \theta_0)) + \sin (m(\theta - \theta_0))}{2} \left( \frac{a}{r_0} \right)^m \]

\[ = -\frac{\lambda}{\pi \epsilon_0 a} \sum_{m=1}^{\infty} \frac{\sin (m\pi) + \sin (0)}{2} \left( \frac{a}{r_0} \right)^m = 0 \]

\[ E_{0, r} = -\frac{\partial \phi}{\partial r} = -\frac{\lambda}{\pi \epsilon_0} \sum_{m=1}^{\infty} \sin (m\theta) \sin (m\theta_0) \frac{r^{m-1}}{r_0^m} = -\frac{\lambda}{\pi \epsilon_0 a} \sum_{m=1}^{\infty} \delta_{\text{odd}, m} \left( \frac{a}{r_0} \right)^m = -\frac{\lambda}{\pi \epsilon_0 r_0} \]

Note that \( E_r = 2E_{0, r} \) implies the addition of the hemisphere doubles the field contribution at the apex of the hemisphere, even in the limit \( \frac{a}{r_0} \ll 1 \). This is consistent with our understanding of fields increasing at sharp boundaries.

**Method of images considerations**

Manipulate the full solution to reveal the images present:

\[ \phi (r, \theta) = \frac{\lambda}{\pi \epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \sin (m\theta_0) \sin (m\theta) \left\{ \left( \frac{r}{r_0} \right)^m - \left( \frac{w_0}{r} \right)^m , a \leq r \leq r_0 \right\} \]

\[ \phi (r, \theta) = \frac{\lambda}{\pi \epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \sin (m\theta_0) \sin (m\theta) \left\{ \left( \frac{r}{r_0} \right)^m - \left( \frac{w_0}{r} \right)^m , r > r_0 \right\} - \frac{\lambda}{\pi \epsilon_0} \sum_{m=1}^{\infty} \frac{\sin (m\theta_0) \sin (m\theta)}{m} \left( \frac{w_0}{r} \right)^m \]

\[ \phi (r, \theta) = \phi_0 (r, \theta) - \frac{\lambda}{\pi \epsilon_0} \sum_{m=1}^{\infty} \frac{\sin (m\theta_0) \sin (m\theta)}{m} \left( \frac{w_0}{r} \right)^m \]

for \( w_0 \equiv \frac{a^2}{r_0} \) and where \( \phi_0 \) refers to the potential in (42), the potential from the original line charge and its mirror image about a conducting ground plane. Note that \( r \geq a > w_0 \), which indicates that the second potential corresponds to two mirror line charges located at \( w_0 \langle \cos \theta_0, \pm \sin \theta_0 \rangle \), with charges \( \mp \lambda \). In total, one can solve the problem with four line charges (1 real and 3 images): charges \( \pm \lambda \) at \( r_0 \langle \cos \theta_0, \pm \sin \theta_0 \rangle \), and charges \( \mp \lambda \) at \( w_0 \langle \cos \theta_0, \pm \sin \theta_0 \rangle \) for \( w_0 \equiv \frac{a^2}{r_0} \).